Exercise 5 - Computational Models - Spring 2012

1. Let L be a language satisfying the following conditions:

   (i) L is a language of TM encodings, i.e. $x \in L$ implies that $x = \langle M \rangle$ for some TM M.

   (ii) L is not trivial, i.e. $x_1 \in L$ and $x_2 \notin L$ for some x_1 and x_2.

   (iii) L is semantic, i.e. if $L(M_1) = L(M_2)$ then $\langle M_1 \rangle \in L$ iff $\langle M_2 \rangle \in L$.

   (a) Use Rice Theorem to prove that $L \notin \mathcal{R}$.

   (b) Do (i) and (ii) suffice to prove that $L \notin \mathcal{R}$? Justify.

2. For the following decision problems determine (and prove your claim) whether they belong to $\mathcal{R}$, $\mathcal{RE} \setminus \mathcal{R}$, co-$\mathcal{RE} \setminus \mathcal{R}$ or none of the above:

   (a) Input: An encoding of a Turing machine M
   Question: is there an x for which M halts?

   (b) Input: An encoding of a Turing machine M
   Question: is there life beyond earth ?

   (c) Input: An encoding of a Turing machine M and inputs x and y
   Question: does M halt on exactly one of the inputs?

   (d) Input: An encoding of a Turing machine M
   Question: is $|L(M)| \geq 3$?

   (e) Input: An encoding of a Turing machine M
   Question: is $L(M) \in \mathcal{RE}$?

   (f) Input: An encoding of a Turing machine M
   Question: does the following holds: $|\langle M \rangle| < 10^{100}$ and M halts on ϵ?

   (g) Input: An encoding of a Turing machine M
   Question: Is it true that for all inputs x, M's run over x never reaches position $|x| + 7$ on the tape?

   (h) Input: An encoding of a Turing machine M and an input x
   Question: does the following holds:
   M's transition function is in $Q \times \Gamma \to Q \times \Gamma \times \{R\}$ (in particular, M can only move right), and M halts on x?

   (i) Input: Encodings of LBAs A_1,A_2
   Question: Does $L(A_1) = L(A_2)$?

   (j) Input: Encodings of DFAs A_1,A_2
   Question: Does $L(A_1) = L(A_2)$?

   (k) Input: Encodings of CFGs A_1,A_2
   Question: Does $L(A_1) = L(A_2)$?

3. Let $L_1, L_2 \in \mathcal{RE} \setminus \mathcal{R}$. Prove whether the following is possible:

   (a) $L_1 \cup L_2 \in \mathcal{R}$

   (b) $L_1 \cup L_2 \in \mathcal{R}$ and $L_1 \cap L_2 \in \mathcal{R}$

4. Let $\emptyset \neq \mathcal{C} \subsetneq \mathcal{RE}$ and let $L = \{\langle M \rangle \mid L(M) \in \mathcal{C}\}$. Suppose that $\langle M_{loopy} \rangle \in L$, where M_{loopy} is a TM that never halts. Is it possible that $L \in \mathcal{RE}$? prove.

5. BONUS. Let $A \in \mathcal{RE} \setminus \mathcal{R}$. Let $B = \{\langle M \rangle \mid A \leq_m L(M)\}$. Determine (and prove your claim) whether B belongs to $\mathcal{R}$, $\mathcal{RE} \setminus \mathcal{R}$, co-$\mathcal{RE} \setminus \mathcal{R}$ or none of the above.