Exercise 2 - Computational Models - Spring 2012

Note1: We denote by $\#_\sigma(w)$ the number of times the word $\sigma \in \Sigma^*$ is a substring in the word $w \in \Sigma^*$.

Note2: You may freely use results from the lectures and recitations.

1. Let L be a language over $\{0, 1, ..., 9\}$ such that $w \in L$ iff w is an i.d. number of a member of the Knesset of 2020. Prove that L is regular.
2. Determine whether the following languages are regular. Prove your answer.
   (a) $L_1 = \{w \mid \#_a(w) = \#_b(w)\}$ over $\Sigma = \{a, b, c\}$.
   (b) $L_2 = \{w \mid |w| \in \mathbb{N}_{even} \ \wedge \ w = w^R\}$ over $\Sigma = \{(,)\}$.
   (c) $L_3 = \{w \mid |w| \in \mathbb{N}_{even} \ \wedge \ w = w^R\}$ over $\Sigma = \{(\}$.
   (d) $L_4 = \{0^m 1^{3m} | m \in \mathbb{N}\}$ over $\Sigma = \{0, 1\}$.
3. Determine whether the following languages are context free. Prove your answer.
   (a) $L_1 = \{a^{2n} b^{3n} | n \in \mathbb{N}\}$ over $\Sigma = \{a, b\}$.
   (b) $L_2 = \{a^n b^{m+n} c^m | n, m \in \mathbb{N}\}$ over $\Sigma = \{a, b, c\}$.
   (c) $L_3 = \{x\#y | x, y \in \{0, 1\}^* \wedge x \ is \ a \ substring \ of \ y\}$ over $\Sigma = \{0, 1, \#\}$.
   (d) $L_4 = \{0^n 1^{n^2} | n \in \mathbb{N}\}$ over $\Sigma = \{0, 1\}$.
   (e) $L_5 = \{w | w = w^R\}$ over $\Sigma = \{0, 1\}$.
   (f) $L_6 = \{w | w \neq w^R\}$ over $\Sigma = \{0, 1\}$.
4. Let $Inv(L) = \{xyz \mid xy^R z \in L\}$. Prove or disprove:
   - The regular languages are closed under this operation.
   - The context free languages are closed under this operation.

5. Define a binary operation on languages over the same alphabet Σ:

$$L_1/L_2 = \{x \in \Sigma^* \mid \exists y \in L_2 . xy \in L_1\}$$

   (a) Let $L_1 = \{0^n 1^n \mid n \in \mathbb{N}\}$ and $L_2 = Ł(1^*)$. What is L_1/L_2?

   (b) Let $L_1 = Ł((01)^*)$ and $L_2 = Ł((0 \cup 1)^* 0)$. What is L_1/L_2?

   (c) Give an example of a context free language L_1 and a language L_2 such that L_1/L_2 is not a context free language.

6. For a binary word w we define $\overline{w}$ as the word with every bit flipped. For example $\overline{010}$ is 101. Let $L = \{w\overline{w} \mid w \in \{0,1\}^*\}$.

   (a) Use the pumping lemma to prove that L is not CFG.

   (b) Find a context free grammar for $\overline{L}$. What does it imply about context free languages?

7. This question deals with algorithmic problems.

   (a) Describe an algorithm that given a DFA A, decides if $L(A)$ is infinite. (possible hint: use the pumping lemma).

   (b) Describe an algorithm that given a DFA A, decides if $|L(A)| = 9,122,009$.