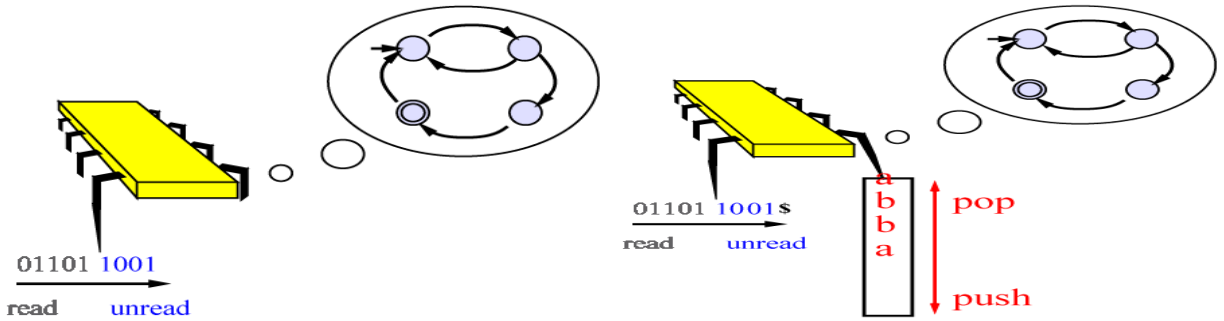
Computational Models - Lectures 4

- Context Free Grammars/Languages (CFG/CFL)
- Algorithmic issues for CFL
- Pumping Lemma for context free languages

Next week:

- Push Down Automata (PDA)
- Equivalence of CFGs and PDAs



- Sipser's book, 2.1, 2.2 & 2.3

Short Overview of the Course

So far we saw

- finite automata,
- regular languages,
- regular expressions,
- Myhill-Nerode theorem
- pumping lemma for regular languages.

We now introduce stronger machines and languages with more expressive power:

- pushdown automata,
- context-free languages,
- context-free grammars,
- pumping lemma for context-free languages.

Context Free Grammars (CFG)

An example of a context free grammar, G_1:

- $\mathbf{A} \rightarrow 0A1$
- $A \rightarrow B$
- $B \rightarrow \#$

Terminology:

- Each line is a substitution rule or production.
- Each rule has the form: symbol $\rightarrow$ string.
  The left-hand symbol is a variable
  (usually upper-case).
- A string consists of variables and terminals.
- One variable is the start variable (lhs of top rule).

Rules for Generating Strings

- Write down the start variable.
- Pick a variable written down in current string and a derivation that starts with that variable.
- Replace that variable with right-hand side of that derivation.
- Repeat until no variables remain.
- Return final string (concatenation of terminals).

Process is inherently non deterministic.

Example

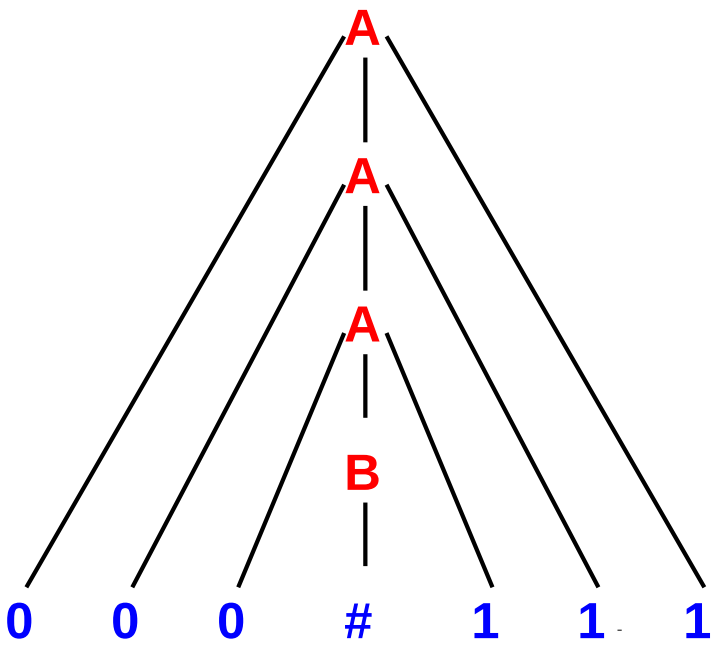
Grammar G_1:

- $A \rightarrow 0A1$
- $A \rightarrow B$
- $B \rightarrow \#$

Derivation with G_1:

$$\begin{aligned} A &\Rightarrow 0A1 \\ &\Rightarrow 00A11 \\ &\Rightarrow 000A111 \\ &\Rightarrow 000B111 \\ &\Rightarrow 000\#111 \end{aligned}$$

A Parse Tree



Indifferent to derivation order.

Question: What strings can be generated in this way from the grammar G_1?
Answer: Exactly those of the form $0^n\#1^n$ $(n \geq 0)$.

Context-Free Languages (CFL)

The language generated in this way is called the language of the grammar.

For example, $L(G_1)$ is $\{0^n\#1^n | n \geq 0\}$.

Any language generated by a context-free grammar is called a context-free language.

A Useful Abbreviation

Rules with same variable on left hand side

$$\begin{aligned} A &\rightarrow 0A1 \\ A &\rightarrow B \end{aligned}$$

are written as:

$$A \rightarrow 0A1 \mid B$$

English-like Sentences

A grammar G_2 to describe a few English sentences:

$$
\begin{aligned}
< \text{SENTENCE} > &\rightarrow < \text{NOUN-PHRASE} >< \text{VERB} > \\
< \text{NOUN-PHRASE} > &\rightarrow < \text{ARTICLE} >< \text{NOUN} > \\
< \text{NOUN} > &\rightarrow \text{boy} \mid \text{girl} \mid \text{flower} \\
< \text{ARTICLE} > &\rightarrow \text{a} \mid \text{the} \\
< \text{VERB} > &\rightarrow \text{touches} \mid \text{likes} \mid \text{sees}
\end{aligned}
$$

Deriving English-like Sentences

A specific derivation in G_2:

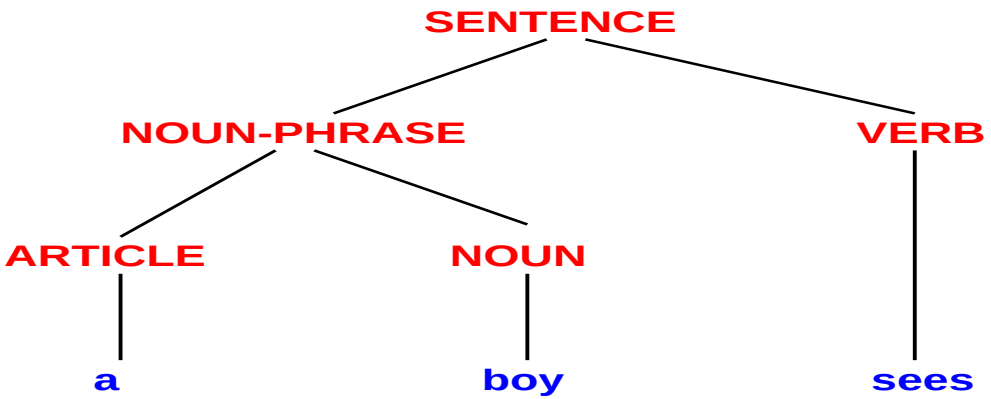
$$
\begin{aligned}
< \text{SENTENCE} > &\Rightarrow < \text{NOUN-PHRASE} >< \text{VERB} > \\
&\Rightarrow < \text{ARTICLE} >< \text{NOUN} >< \text{VERB} > \\
&\Rightarrow \text{a } < \text{NOUN} >< \text{VERB} > \\
&\Rightarrow \text{a boy } < \text{VERB} > \\
&\Rightarrow \text{a boy sees}
\end{aligned}
$$

More strings generated by G_2:

a flower sees
the girl touches

Derivation and Parse Tree

$$
\begin{aligned}
< \text{SENTENCE} > &\Rightarrow < \text{NOUN-PHRASE} >< \text{VERB} > \\
&\Rightarrow < \text{ARTICLE} >< \text{NOUN} >< \text{VERB} > \\
&\Rightarrow \text{a} < \text{NOUN} >< \text{VERB} > \\
&\Rightarrow \text{a boy} < \text{VERB} > \\
&\Rightarrow \text{a boy sees}
\end{aligned}
$$



Formal Definitions

A context-free grammar is a 4-tuple (V, Σ, R, S) where

- V is a finite set of variables,
- Σ is a finite set of terminals,
- R is a finite set of rules: each rule is a variable and a finite string of variables and terminals.
- S is the start symbol.

Formal Definitions (2)

- If u and v are strings of variables and terminals,
- and $A \to w$ is a rule of the grammar, then
- we say uAv yields uwv, written $uAv \Rightarrow uwv$.

We write $u \stackrel{*}{\Rightarrow} v$ if $u = v$ or

$$u \Rightarrow u_1 \Rightarrow \ldots \Rightarrow u_k \Rightarrow v.$$

for some sequence $u_1, u_2, \ldots, u_k$.

Definition: The language of the grammar is

$$\left\{ w \in \Sigma^* \mid S \stackrel{*}{\Rightarrow} w \right\} .$$

Example 1

$G_4 = (\{S\}, \{a, b\}, R, S)$.

R (Rules): $S \to aSb \mid SS \mid \varepsilon$.

Some words in the language: $aabb$, $aababb$.

Q.: But what is this language?

Hint: Think of parentheses.

Example 2

$G_5 = (\{S\}, \{a, b\}, R, S)$.

R (Rules): $S \to aSa \mid bSb \mid \varepsilon$

Some words in the language: $abba$, $aabaabaa$.

Q.: But what is this language?

$L(G_5) = \{ww^R | w \in \{a, b\}^*\}$

Example 3

$G_6 = (\{S, A, B\}, \{a, b\}, R, S)$.

R (Rules):

$S \to aB \mid bA \mid \varepsilon$

$A \to a \mid aS \mid bAA$

$B \to b \mid bS \mid aBB$

Some words in the language: $aababb$, $baabba$.

Q.: But what is this language?

$L(G_6) = \{w \in \{a, b\}^* | \#_a(w) = \#_b(w)\}$

Proving $L(G_6) = \{w \in \{a,b\}^* \,|\#_a(w) = \#_b(w)\}$

- What do we need to show?!
- (1) If w generated by G_6 then $\#_a(w) = \#_b(w)$.
- (2) If $\#_a(w) = \#_b(w)$ then w generated by G_6.
- Claim: If $S \stackrel{*}{\Rightarrow} \alpha$ then $\#_a(\alpha) + \#_A(\alpha) = \#_b(\alpha) + \#_B(\alpha)$.
- Claim: For $w \in \{a,b\}^*$, let $k = \#_a(w) - \#_b(w)$. Then:
  - i. If $k = 0$, then $S \stackrel{*}{\Rightarrow} wS$
  - ii. If $k > 0$, then $S \stackrel{*}{\Rightarrow} wB^k$
  - iii. If $k < 0$, then $S \stackrel{*}{\Rightarrow} wA^k$

  Proof by induction on $|w|$
- Are we done?!

Proving $L(G_6) = \{w \in \{a,b\}^* \,|\#_a(w) = \#_b(w)\}$

- Proof by induction on $|w|$.
- Basis, $w = \epsilon$, then $k = 0$ and $S \stackrel{*}{\Rightarrow} \epsilon S = S$.
- Induction step. Let $w = \sigma_1 \ldots \sigma_n$. Then $w = w'\sigma$ and $|w'| = n - 1$.
- For example suppose that $\#_a(w') - \#_b(w') = k > 0$ and $\sigma = a$.
- By the induction hypothesis $S \stackrel{*}{\Rightarrow} w'B^k$,
- Since $B \Rightarrow aBB$, then $S \stackrel{*}{\Rightarrow} wB^{k+1}$
- To complete the proof, need to show for $k = 0$ and $k < 0$, and also for $\sigma = b$.

Designing Context-Free Grammars

No recipe in general, but few rules-of-thumb

- If CFG is the union of several CFGs, rename variables (not terminals) so they are disjoint, and add new rule $S \rightarrow S_1 \mid S_2 \mid \ldots \mid S_i$.
- For languages (like $\{0^n \# 1^n | n \geq 0\}$), with linked substrings, a rule of form $R \rightarrow uRv$ is helpful to force desired relation between substrings.
- For a regular language, grammar "follows" a DFA for the language (see next slide).

CFG for Regular Languages

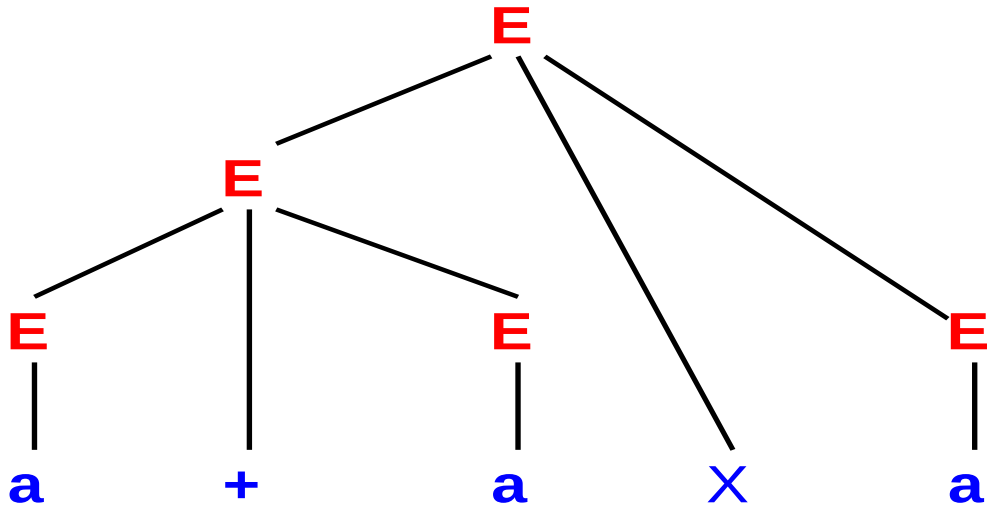
DFA: $M = (Q, \Sigma, \delta, q_0, F)$
CFG for $L(M)$:
R_0 is the staring variable.

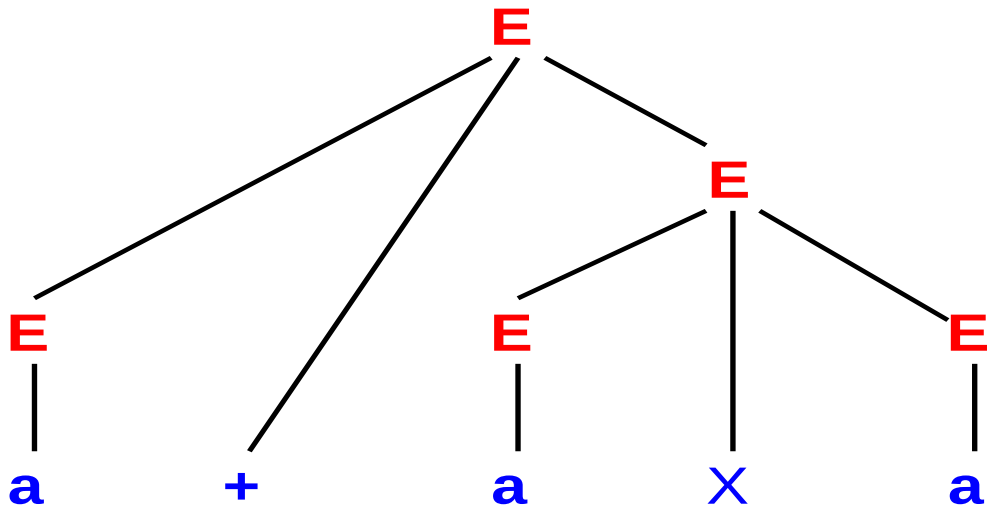
$$\begin{aligned} R_i &\rightarrow aR_j, \text{ if } \delta(q_i, a) = q_j \\ R_i &\rightarrow \varepsilon, \text{ if } q_i \in F \end{aligned}$$

Next class we'll see alternative proof via "Push-Down Automata"

Ambiguity in CFLs

Grammar: $E \rightarrow E{+}E \mid E{\times}E \mid (E) \mid a$





Arithmetic Example

Consider (V, Σ, R, E), where

- $V = \{E, T, F\}$
- $\Sigma = \{a, +, \times, (,)\}$

Rules:

$$\begin{aligned} E &\rightarrow E + T \mid T \\ T &\rightarrow T \times F \mid F \\ F &\rightarrow (E) \mid a \end{aligned}$$

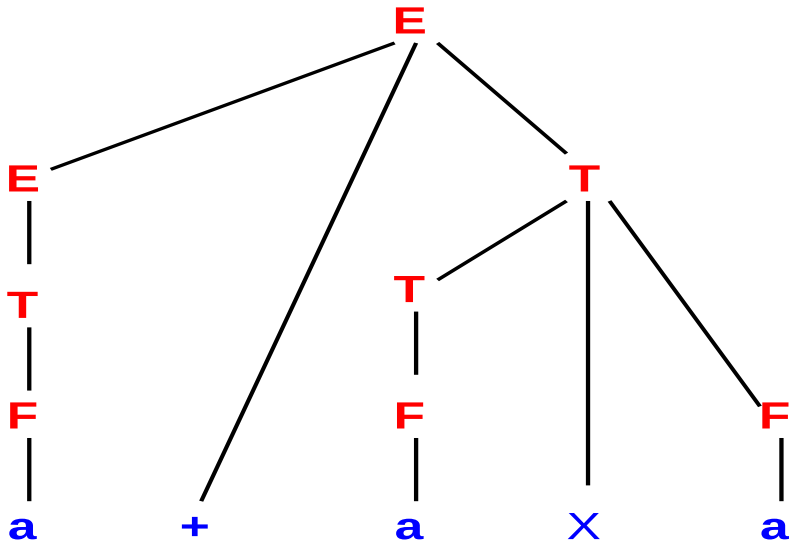
Some strings generated by the grammar:
$a + a \times a$ and $(a + a) \times a$.

Same arithmetic expressions, just not ambiguous.

Proof by induction on the string lenght.

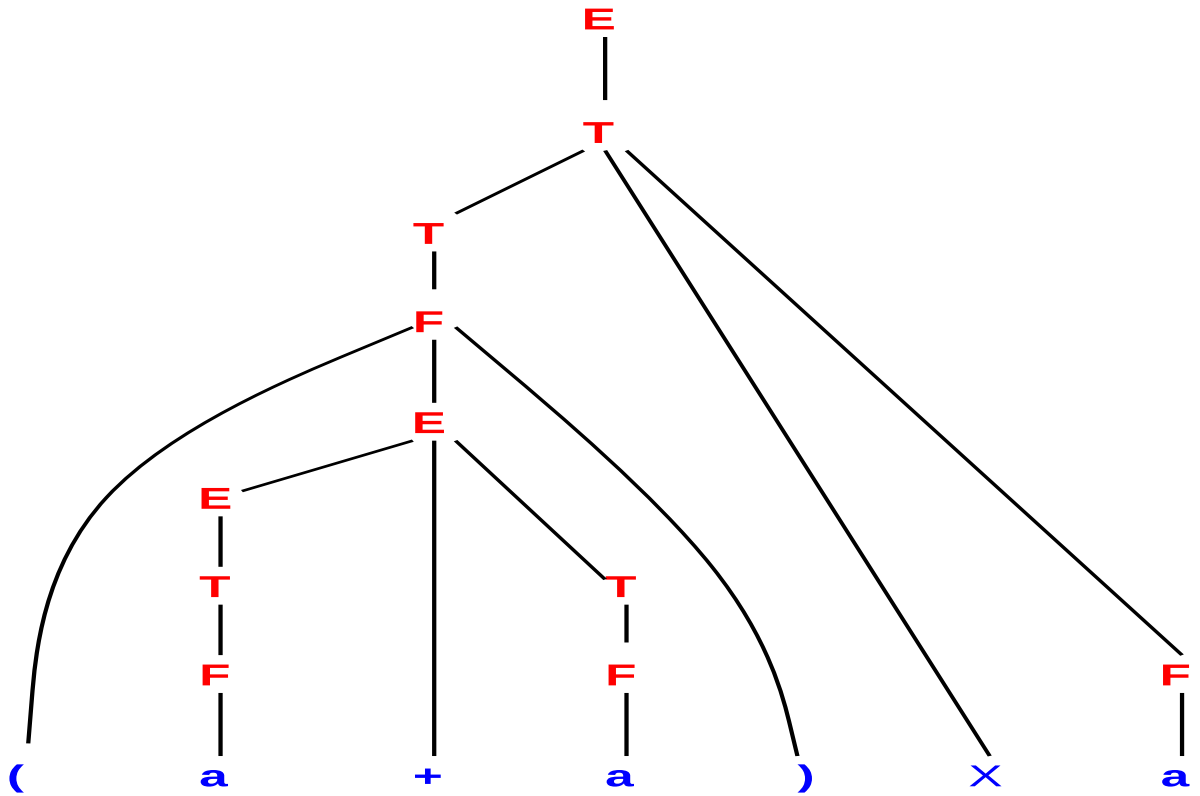
Parse Tree for $a + a \times a$

$$
\begin{aligned}
E &\rightarrow E + T \mid T \\
T &\rightarrow T \times F \mid F \\
F &\rightarrow (E) \mid a
\end{aligned}
$$



Parse Tree for $(a + a) \times a$

$$
\begin{aligned}
E &\rightarrow E + T \mid T \\
T &\rightarrow T \times F \mid F \\
F &\rightarrow (E) \mid a
\end{aligned}
$$



Ambiguity

We say that a string w is derived ambiguously from grammar G if w has two or more parse trees that generate it from G.

Ambiguity is usually not only a syntactic notion but also semantic, implying multiple meanings for the same string. Think of $a + a \times a$ from last grammar.

It is sometime possible to eliminate ambiguity by finding a different context free grammar generating the same language. This is true for the arithmetic expressions grammar.

Some languages, *e.g.* $\{1^i 2^j 3^k \mid i = j \text{ or } j = k\}$ are inherently ambiguous. Proof?

Checking Membership

Checking Membership in a CFL

Given a CFG, G, and a string w, does G generate w?

Initial Idea: Design an algorithm that tries all derivations.

Problem: If G does not generate w, we'll never stop.

Chomsky Normal Form

A simplified, canonical form of context free grammars.
Every rule has the form

$$\begin{aligned} A &\rightarrow BC \\ A &\rightarrow a \\ S &\rightarrow \varepsilon \end{aligned}$$

- S is the start symbol
- B and C are not the start symbol
- A is any variable, and a is any terminal

Simpler to analyze: each derivation adds (at most) a single terminal, S only appears once, ε appears only at the empty word

CNF: Theorem

Theorem: Any context-free language is generated by a context-free grammar in Chomsky Normal Form.

Basic Idea:

- Add new start symbol S_0.
- Eliminate all ε rules of the form $A \to \varepsilon$.
- Eliminate all "unit" rules of the form $A \to B$.
- Patch up rules so that grammar generates the same language.
- Convert remaining "long rules" to proper form.

CNF: Example

$$
\begin{aligned}
S &\rightarrow ASA \mid aB \\
A &\rightarrow B \mid S \\
B &\rightarrow b \mid \varepsilon
\end{aligned}
$$

Transformed into:

$$
\begin{aligned}
S_0 &\rightarrow AA_1 \mid UB \mid a \mid SA \mid AS \\
S &\rightarrow AA_1 \mid UB \mid a \mid SA \mid AS \\
A &\rightarrow b \mid AA_1 \mid UB \mid a \mid SA \mid AS \\
A_1 &\rightarrow SA \\
U &\rightarrow a \\
B &\rightarrow b
\end{aligned}
$$

CNF: Bounded Derivation Length

Lemma: If G is in Chomsky Normal Form, $|w| = n$, and w is generated by G, then w has a derivation of length $2n - 1$. Proof?

Advantage: Easier to check whether $w \in L(G)$

Checking Membership for Grammars in CNF Form

Given a Chomsky normal form grammar $G = (V, \Sigma, R, S)$,

we build a function Derive(A, x) that returns TRUE if $A \stackrel{*}{\Rightarrow} x$.

Function Derive(A, x):

- If $x = \varepsilon$, then if $A \to \varepsilon \in R$ (i.e., $A = S$) return TRUE, otherwise return FALSE.
- If $|x| = 1$ then if $A \to x \in R$ return TRUE, otherwise return FALSE.
- For each $A \to BC$ and each partition $x = x_1 x_2$,
  - Call Derive(B, x_1) and Derive(C, x_2).
  - If both return true, exit and return TRUE.
- Return FALSE.

Test whether $w \in L(G)$ by calling Derive(S, w).

- Correctness?

Checking Membership for Grammars in CNF Form, cont

- Procedure Derive can also output a parse tree for w
- Have we critically used that G is in CNF?

Time Complexity

- What is the time complexity of Derive(S, w) ?
- Simple Recursive implementation analysis:
  - each time test $|R|$ rules and n partitions.
  - $T(n) \leq |R|n \cdot 2T(n-1)$
  - $T(n) = O((|R|n)^n)$.
- Still exponential

Efficient Algorithm

- Keep in memory the results of Derive(A, x).
  - Number of different inputs: $|V| \cdot n^2$.
  - Only $|V| \cdot n^2$ calls, each takes $O(n|R|)$.
  - $T(n) = O(|R|n^3|V|)$.
- Polynomial time!

Dynamic Programming

- This approach is called Dynamic Programming
- Basic idea:
- If number of different inputs is limited, say $I(n)$.
- Each run (excluding recursive calls) takes at most $R(n)$ time
- Total running time is bounded by $T(n) \leq R(n)I(n)$.

Non-Context-Free Languages

Proving a Language is not a CFL

- The pumping lemma for finite automata and Myhill-Nerode theorem are our tools for showing that languages are not regular.
- We will now show a similar pumping lemma for context-free languages.
- It is slightly more complicated . . .

Pumping Lemma for CFL

Also known as the $uvxyz$ Theorem.

Theorem: If L is a CFL , there is an ℓ (critical length), such that if $w \in L$ and $|w| \geq \ell$, then $w = uvxyz$ where

- for every $i \geq 0$, $uv^i xy^i z \in L$
- $|vy| > 0$, (non-triviality)
- $|vxy| \leq \ell$.

Basic Intuition

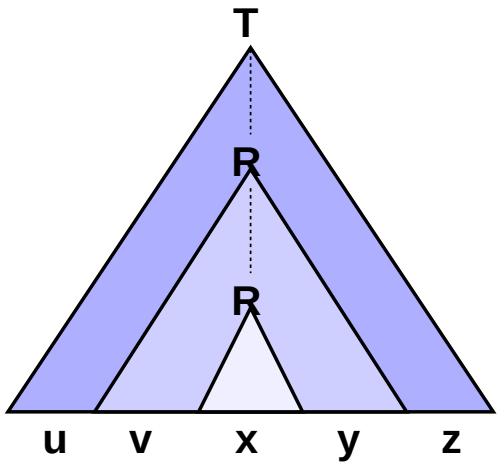
Let L be a CFL and G its CFG.

Let w be a "very long" string in L.

Then w must have a "tall" parse tree.

And some root-to-leaf path must repeat a symbol.

ahmmm,. . . why is that so?



We have: $T \stackrel{*}{\Rightarrow} uRz$, $R \stackrel{*}{\Rightarrow} vRy$, and $R \stackrel{*}{\Rightarrow} x$.

Proof

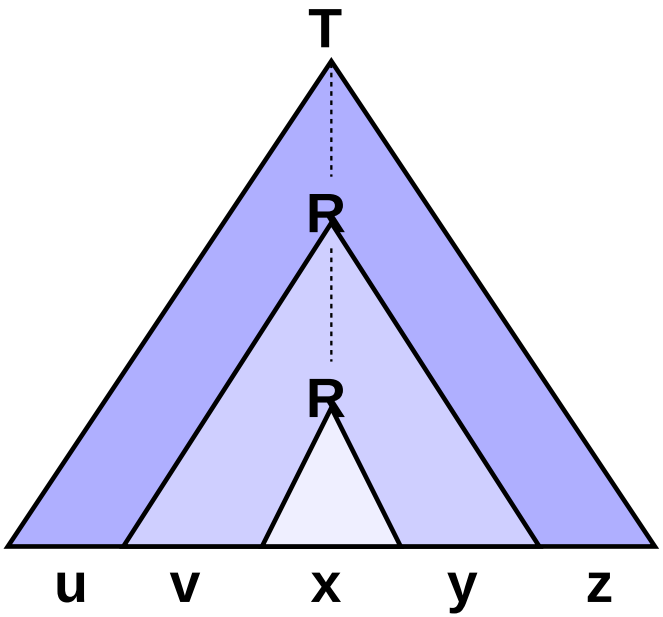
- let G be a CFG for CFL L.
- let b be the max number of symbols in right-hand-side of any rule (what is b for CNF grammars?).
- no node in parse tree has $> b$ children.
- at depth d, can have at most b^d leaves.
- let $|V|$ be the number of variables in G.
- set $\ell = b^{|V|+2}$.

Proof (2)

- let w be a string with $|w| \geq \ell$
- let T be parse tree for w with fewest nodes
- T has height $\geq |V| + 2$
- some path in T has length $\geq |V| + 2$
- that path repeats a variable R

Proof (3)

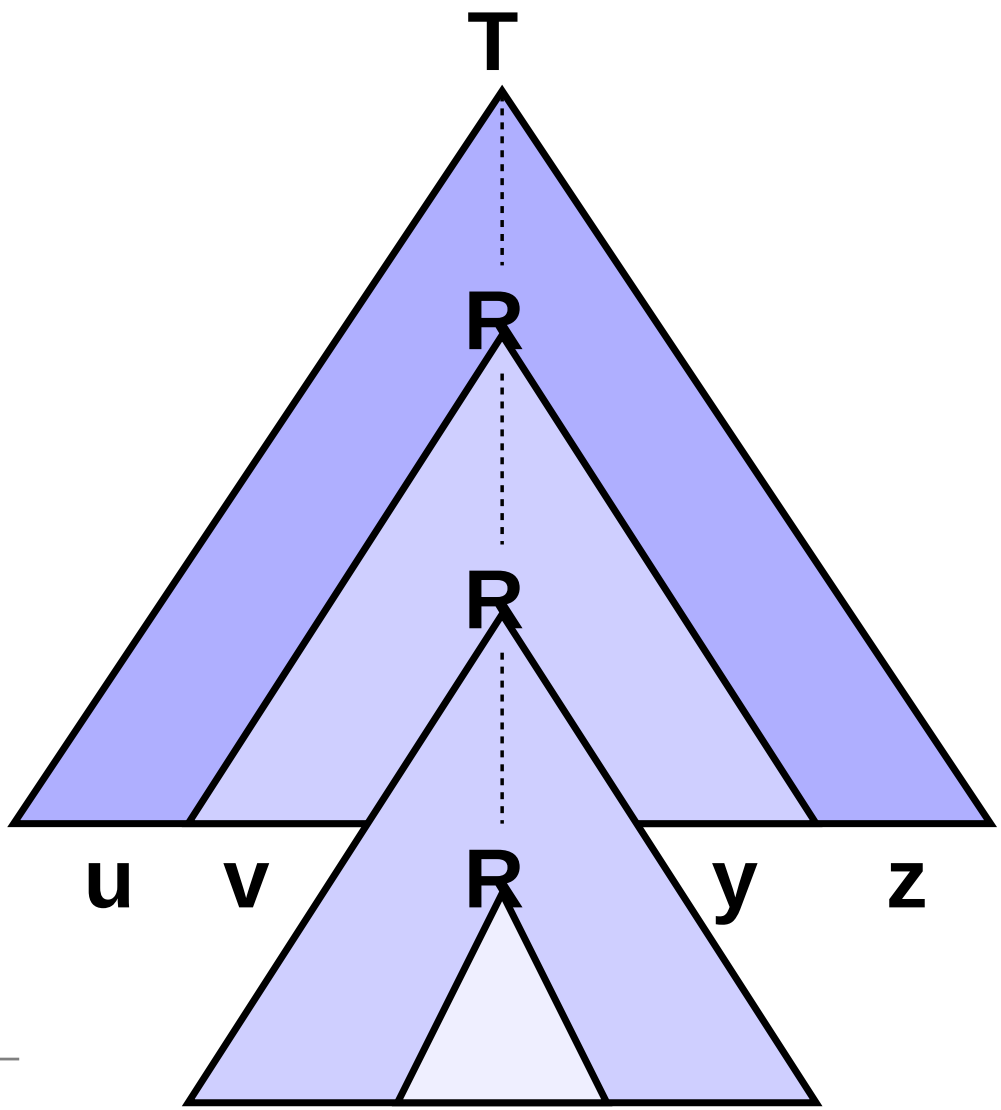
Split $w = uvxyz$



- each occurrence of R produces a string
- upper produces string vxy
- lower produces string x

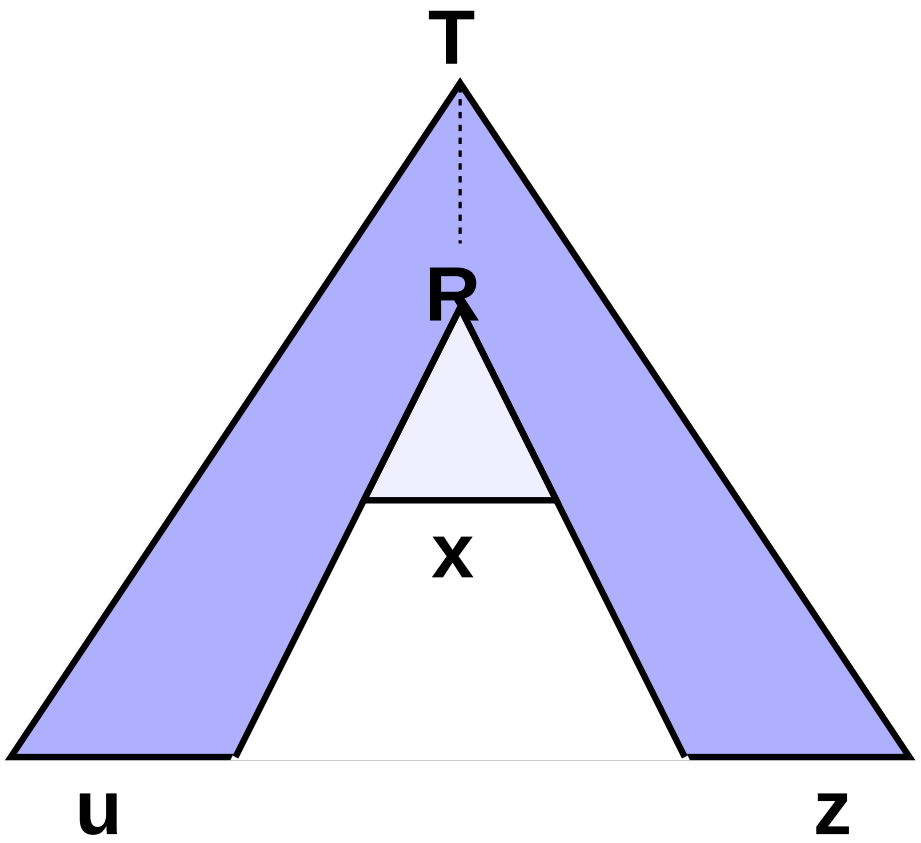
Proof

Replacing smaller by larger yields $uv^i xy^i z$, for $i > 0$.



Proof (4)

Replacing larger by smaller yields uxz.



Together, they establish:

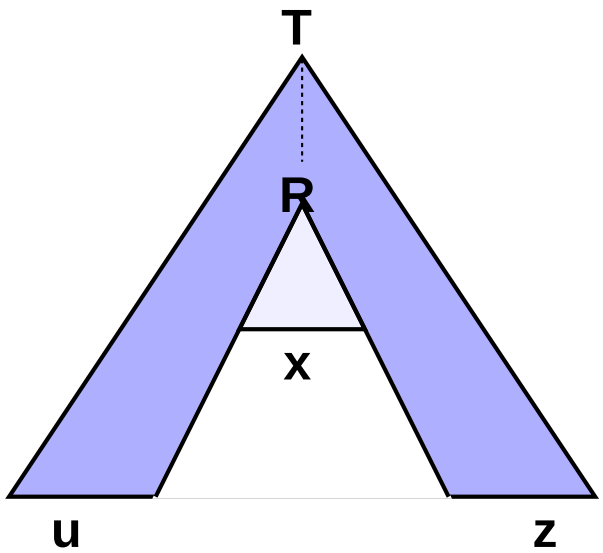
- for all $i \geq 0$, $uv^ixy^iz \in L$

Proof (5)

Next condition is:

- $|vy| > 0$ (out of $uvxyz$)

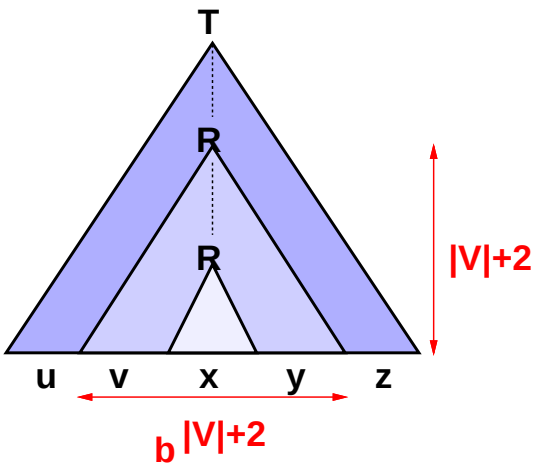
If v and y are both ε, then



is a parse tree for w with fewer nodes than T, contradiction.

Proof (6)

Final condition is $|vxy| \leq \ell$:



- the upper occurrence of R generates vxy.
- can choose symbols such that both occurrences of R lie in bottom $|V| + 1$ variables on path.
- subtree where R generates vxy is $\leq |V| + 2$ high.
- strings by subtree at most $b^{|V|+2} = \ell$ long. ♠

Non CFL Example (1)

Claim: $B = \{a^n b^n c^n\}$ is not a CFL.
Proof: By contradiction.

Let ℓ be the critical length, and consider $w = a^\ell b^\ell c^\ell$.
If $w = uvxyz$, neither v nor y can contain

- both a's and b's, or
- both b's and c's,

because otherwise uv^2xy^2z would have out-of-order symbols.

But if v and y contain only one letter, then uv^2xy^2z has an imbalance!

Non CFL Example (2)

Claim: $C = \{a^i b^j c^k | 0 \leq i \leq j \leq k\}$ is not context free.

Let ℓ be the critical length, and $w = a^\ell b^\ell c^\ell$.

Let $w = uvxyz$

- Neither v nor y contains two distinct symbols, because otherwise uv^2xy^2z would have out-of-order symbols.
- vxy cannot be all the same letter (why?)
- $|vxy| \leq \ell$, so either
  - v contains only a's and y contains only b's, but then uv^2xy^2z has too few c's.
  - v contains only b's and y contains only c's, but then uv^0xy^0z has too many a's.

Non CFL Example (3)

Claim: $D = \{ww | w \in \{0,1\}^*\}$ is not context-free.

Let $w = 0^\ell 1^\ell 0^\ell 1^\ell$. As before , suppose $w = uvxyz$

Recall that $|vxy| \leq \ell$.

- if vxy is in the first half of w, uv^2xy^2z “moves" a 1 into the first position in second half.
- if vxy is in the second half, uv^2xy^2z “moves" a 0 into the last position in first half.
- if vxy straddles the midpoint, then pumping *down* to uxz yields $0^\ell 1^i 0^j 1^\ell$ where i and j cannot both be ℓ.

Note that $D = \{ww^R | w \in \{0,1\}^*\}$ is CFL.