Administrative Notes

- MidTerm Exam: Tentative Friday May 4th.
- Homework assignment 1 was published last week.
  - It is unlikely you'll be able to solve it on your own if the first time you think about is the night before the deadline.
  - While we can hardly detect dependencies in preparation of the homework (still, this sometimes happens), we actively enforce mutual independence in the midterm and final exams.

Computational Models - Lecture 2

- Non-Deterministic Finite Automata (NFA)
- Closure of Regular Languages Under $\bigcup, \circ, \; ^*$
- Regular expressions
- Equivalence with finite automata
  - Sipser's book, 1.1-1.3

DFA Formal Definition (reminder)

A deterministic finite automaton (DFA) is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

- Q is a finite set called the states,
- Σ is a finite set called the alphabet,
- $\delta : Q \times \Sigma \to Q$ is the transition function,
- $q_0 \in Q$ is the start state, and
- $F \subseteq Q$ is the set of accept states.

Languages and DFA (reminder)

Definition: Let L ($L \subseteq \Sigma^*$) be the set of strings that M accepts. $L(M)$, the language of a DFA M, is defined as $L(M) = L$.

Note that

- M may accept many strings, but
- M accepts only one language.

A language is called regular if some deterministic finite automaton accepts it.

The Regular Operations (reminder)

Let A and B be languages.

The union operation:

$$A \cup B = \{x | x \in A \text{ or } x \in B\}$$

The concatenation operation:

$$A \circ B = \{xy | x \in A \text{ and } y \in B\}$$

The star operation:

$$A^* = \{x_1 x_2 \ldots x_k | k \geq 0 \text{ and each } x_i \in A\}$$

Claim: Closure Under Union (reminder)

If A_1 and A_2 are regular languages, so is $A_1 \cup A_2$.

Approach to Proof:

- some M_1 accepts A_1
- some M_2 accepts A_2
- construct M that accepts $A_1 \cup A_2$.
- in our construction, states of M were Cartesian product of M_1 and M_2 states.

What About Concatenation?

Thm: If L_1, L_2 are regular languages, so is $L_1 \circ L_2$.

Example: $L_1 = \{\text{good}, \text{bad}\}$ and $L_2 = \{\text{boy}, \text{girl}\}$.

$$L_1 \circ L_2 = \{\text{goodboy}, \text{goodgirl}, \text{badboy}, \text{badgirl}\}$$

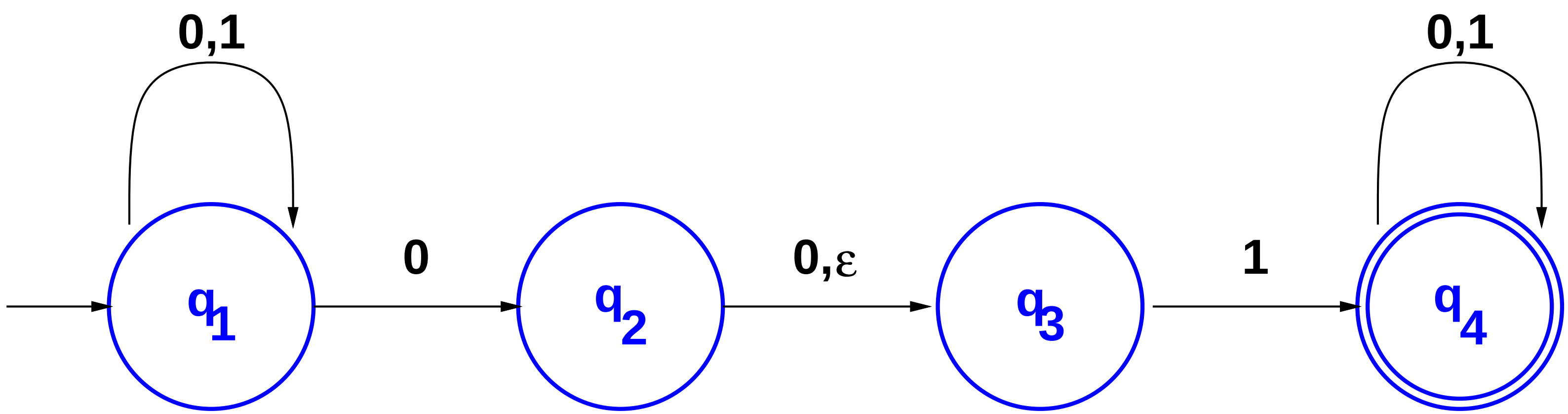
This is much harder to prove.

Idea: Simulate M_1 for a while, then switch to M_2.

Problem: But when do you switch?

Seems hard to do with DFAs.
This leads us into non-determinism.

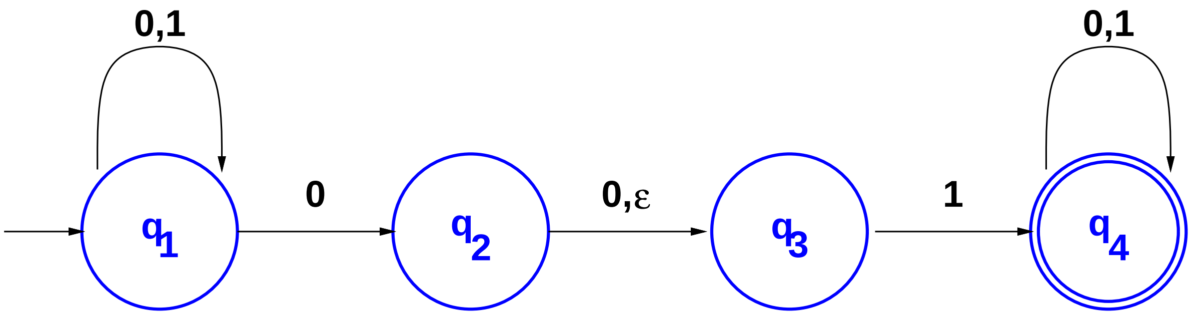
Non-Deterministic Finite Automata



- an NFA may have more than one transition labeled with the same symbol,
- an NFA may have no transitions labeled with a certain symbol, and
- an NFA may have transitions labeled with ε, the symbol of the empty string.

Comment: Every DFA is also a non-deterministic finite automata (NFA).

Non-Deterministic Computation

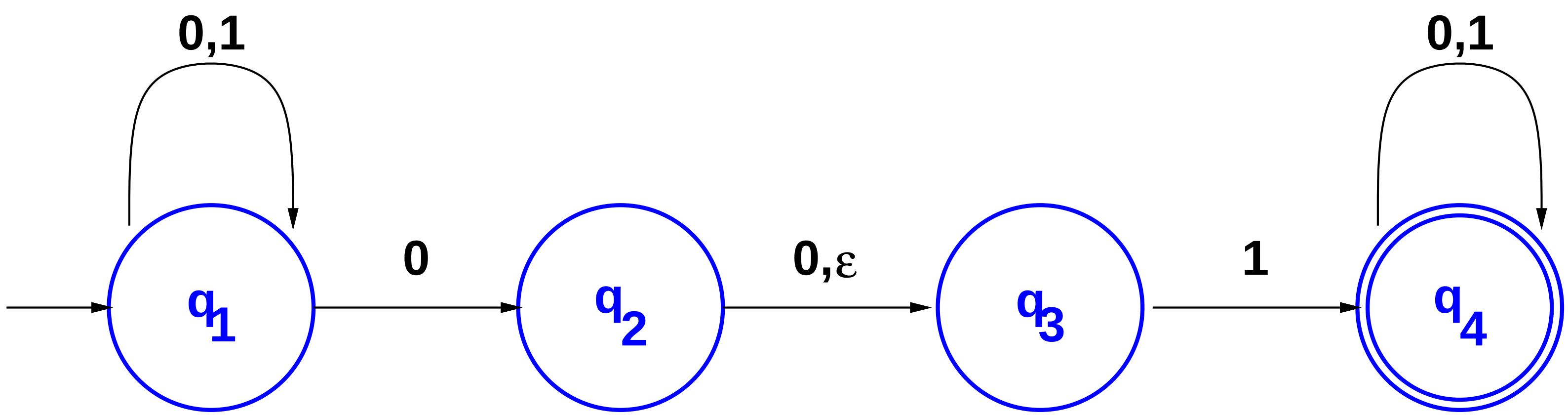


What happens when more than one transition is possible?

- the machine "splits" into multiple copies
- each branch follows one possibility
- together, branches follow all possibilities.
- If the input doesn't appear, that branch "dies".
- Automaton accepts if some branch accepts.

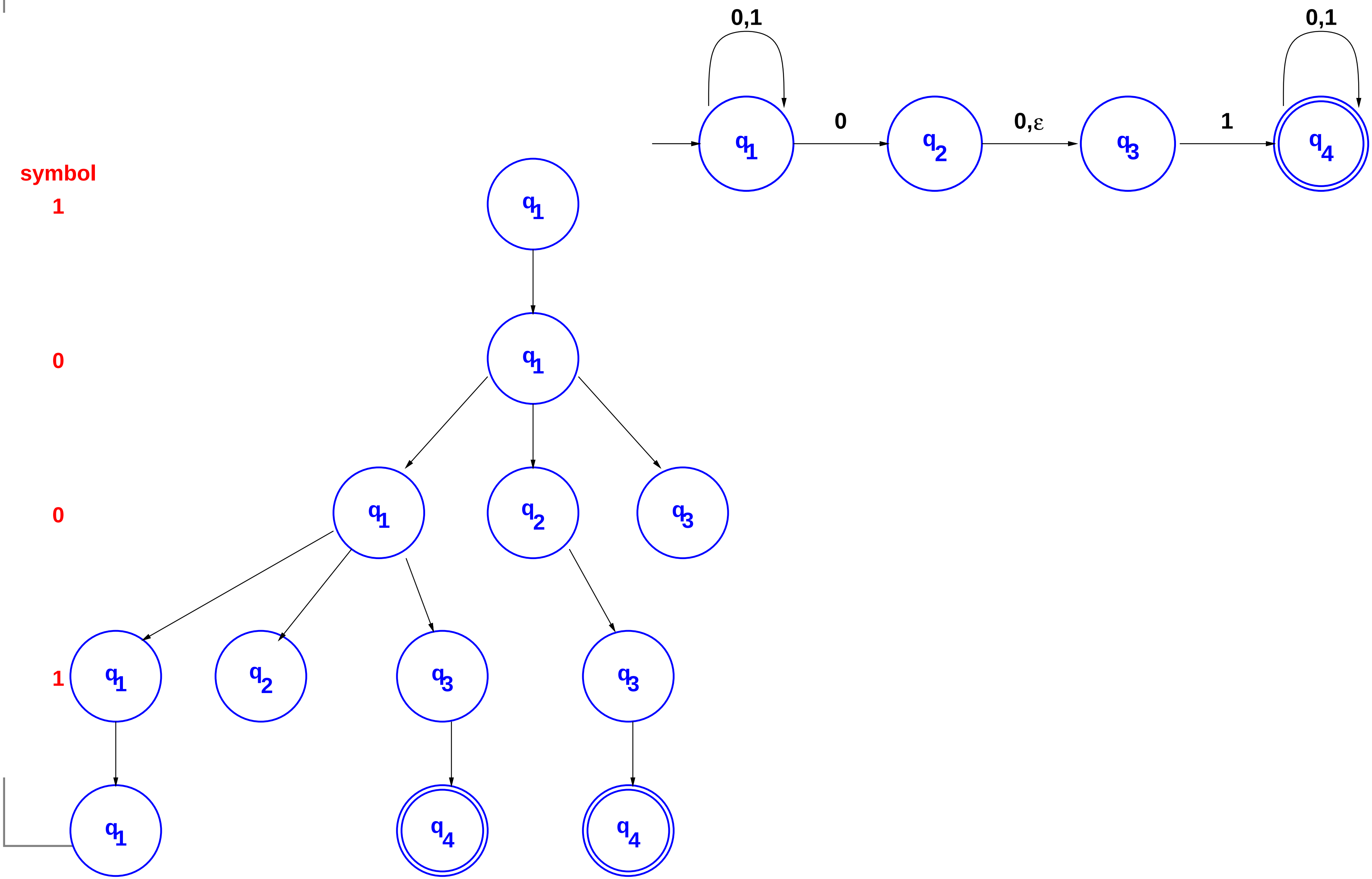
What does an ε transition do?

Non-Deterministic Computation



What happens on string 1001?

The String 1001



Why Non-Determinism?

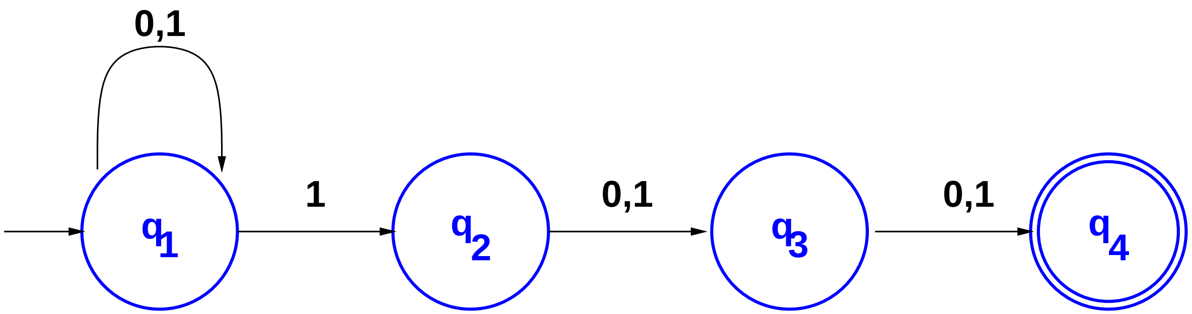
Theorem (to be proved soon): Deterministic and non-deterministic finite automata accept exactly the same set of languages.

Q.: So why do we need them?

A.: NFAs are often easier to design than equivalent DFAs.

Example: Design a finite automaton that accepts all strings with a 1 in their third-to-the-last position?
NFA ?
DFA ?

Solving with NFA

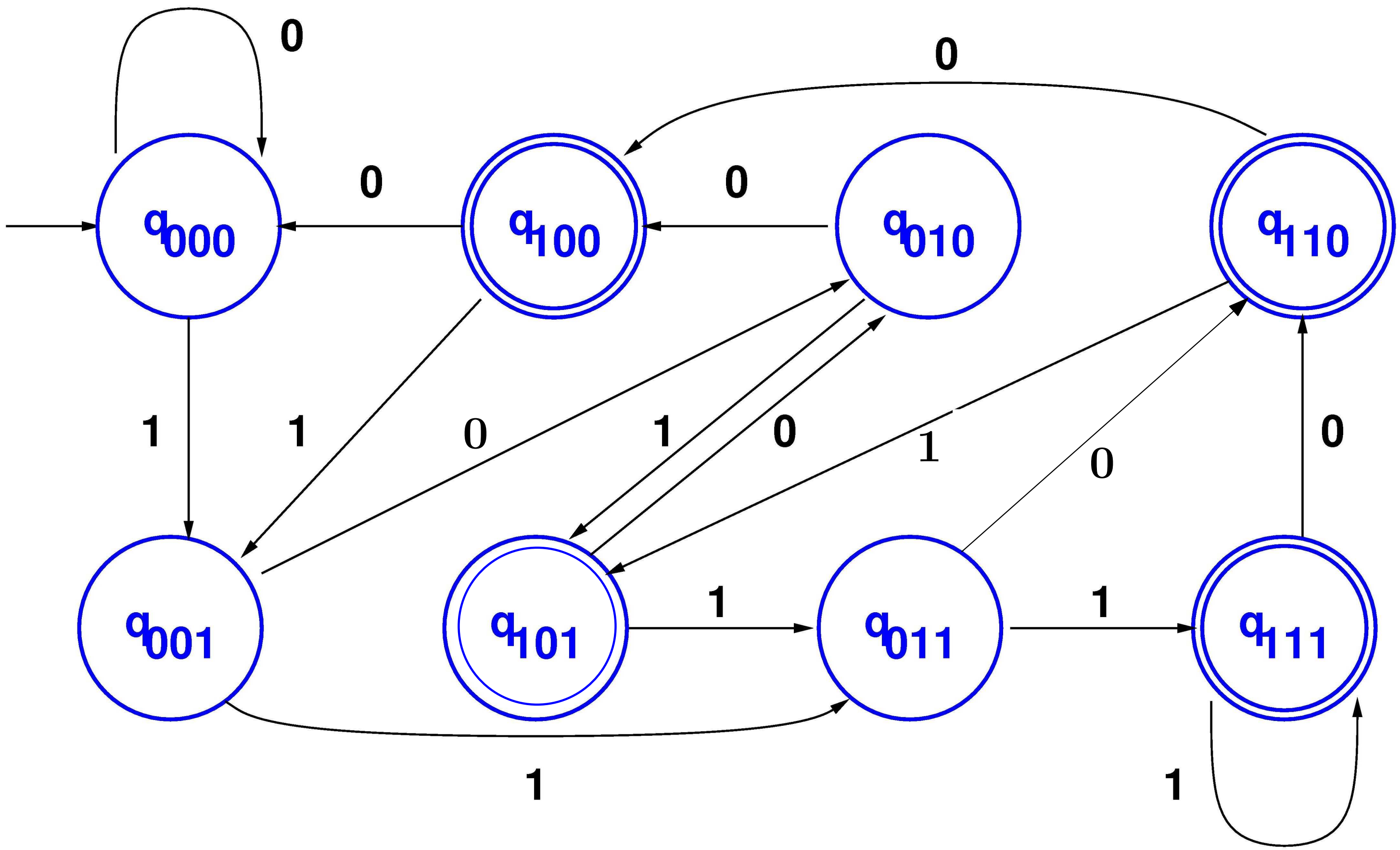


- “Guesses” which symbol is third from the last, and
- checks that indeed it is a 1.
- If guess is premature, that branch “dies”, and no harm occurs.

Solving with DFA

- Have 8 states, encoding the last three letters.
  - A state for each string in $\{0,1\}^3$.
- add transitions on modifying the suffix, give the new letter.
- Mark as accepting, the strings $1**$

Solving with DFA



NFA – Formal Definition

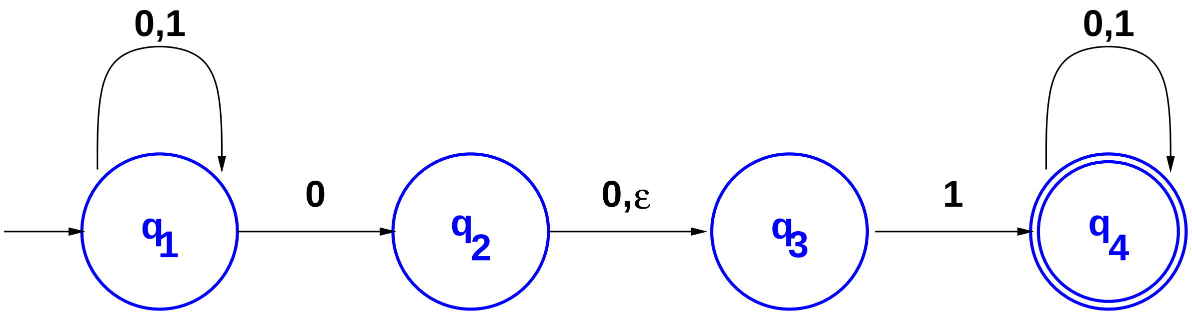
Transition function δ is going to be different.

- Let $\mathcal{P}(Q)$ denote the powerset of Q.
- Let Σ_ε denote $\Sigma \cup \{\varepsilon\}$.

A non-deterministic finite automaton is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

- Q is a finite set called the states,
- Σ is a finite set called the alphabet,
- $\delta : Q \times \Sigma_\varepsilon \to \mathcal{P}(Q)$ is the transition function,
- $q_0 \in Q$ is the start state, and
- $F \subseteq Q$ is the set of accept states.

Example



$N_1 = (Q, \Sigma, \delta, q_1, F)$

where

- $Q = \{q_1, q_2, q_3, q_4\}$, $\Sigma = \{0, 1\}$,
- δ is

	0	1	ε
q_1	$\{q_1, q_2\}$	$\{q_1\}$	$\emptyset$
q_2	$\{q_3\}$	$\emptyset$	$\{q_3\}$
q_3	$\emptyset$	$\{q_4\}$	$\emptyset$
q_4	$\{q_4\}$	$\{q_4\}$	$\emptyset$

- q_1 is the start state, and $F = \{q_4\}$.

Formal Model of Computation

- Let $M = (Q, \Sigma, \delta, q_0, F)$ be an NFA, and
- $y = y_1 y_2 \cdots y_m$ be a string over Σ_ε.
- u be the string over Σ obtained from y by omitting all occurances of ε.

Suppose there is a sequence of *states* (in Q), $r_0, \ldots, r_n$, such that

- $r_0 = q_0$
- $r_{i+1} \in \delta(r_i, y_{i+1})$, $0 \leq i < n$
- $r_n \in F$

Then we say that M accepts u.
Does M accept the empty string?

Equivalence of NFA's and DFA's

- Given an NFA, N, we construct a DFA, M, that accepts the same language.
- To begin with, we make things easier by ignoring ε transitions.
- Make DFA simulate all possible NFA states.
- As consequence of the construction, if the NFA has k states, the DFA has 2^k states (an exponential blow up).

Equivalence of NFA's and DFA's

Let $N = (Q, \Sigma, \delta, q_0, F)$ be the NFA accepting A.

Construct a DFA $M = (Q', \Sigma, \delta', q'_0, F')$.

- $Q' = \mathcal{P}(Q)$.
- For $R \in Q'$ and $a \in \Sigma$, let

$$\delta'(R, a) = \bigcup_{r \in R} \delta(r, a) = \{q \in Q | q \in \delta(r, a) \text{ for some } r \in R\}$$

- $q'_0 = \{q_0\}$
- $F' = \{R \in Q' | R \text{ contains an accept state of } N\}$

Notice: F' is a set whose elements are subsets of Q, so (as expected) F' is a subset of $\mathcal{P}(Q)$.

Equivalence of NFA's and DFA's - proof

- Extend the transition function to work on words:
  $\delta'(q, w_1 \cdots w_n) = \delta'(\delta'(q, w_1 \cdots w_{n-1}), w_n)$,
  $\delta(q, w_1 \cdots w_n) = \bigcup_{q \in \delta(q, w_1 \cdots w_{n-1})} \delta(q, w_n)$, ($\delta'(q, \epsilon) = q$)
- Define an equivalence between subset of states $R \subset Q$ in the NFA and the states $q(R) \in Q'$ in DFA
- How do we proceed with a proof ?!

Equivalence of NFA's and DFA's - proof

- Inductive Claim: For any word $y = y_1 y_2 \cdots y_m \in \Sigma^*$ of length m, $\delta(q_0, y)$ is equivalent to $\delta'(q_0, y)$.
- Proof by induction on m.
- Base of the induction: $\delta(q_0, \epsilon) = q_0$ and $\delta'(\{q_0\}, \epsilon) = \{q_0\}$.
- Inductive step. Fix $y = y_1 \ldots y_m$ and let $R_{m-1} = \delta'(q_0, y_1 \ldots y_{m-1})$

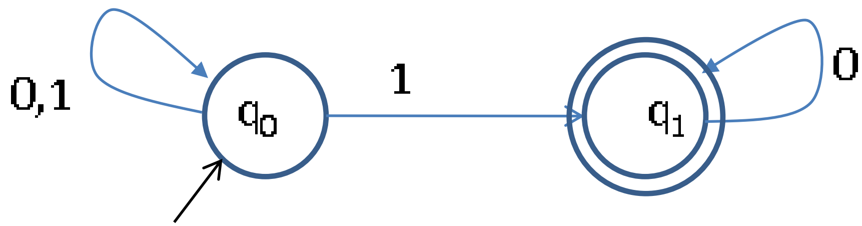
$$\delta'(q_0, y) = \delta'(R_{m-1}, y_m) = \bigcup_{r \in R_{m-1}} \delta(r, y_m)$$

$$= \bigcup_{r \in \delta(q_0, y_1 \ldots y_{m-1})} \delta(r, y_m) = \delta(q_0, y)$$

- Example.

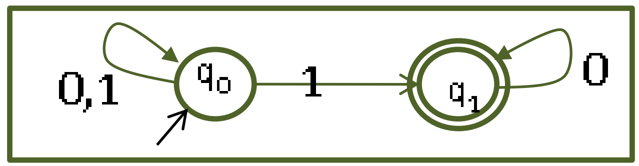
Example: NFA $\Rightarrow$ DFA

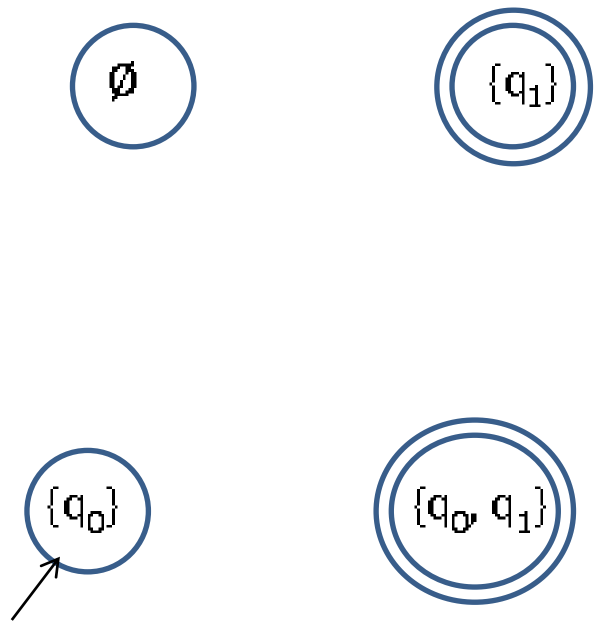
Non-Deterministic Automata:



Example: NFA $\Rightarrow$ DFA

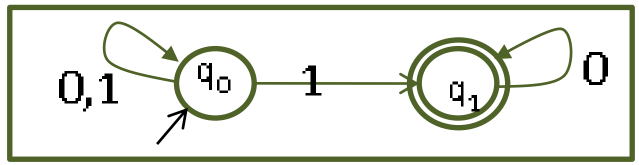
Deterministic Automata - set of states:

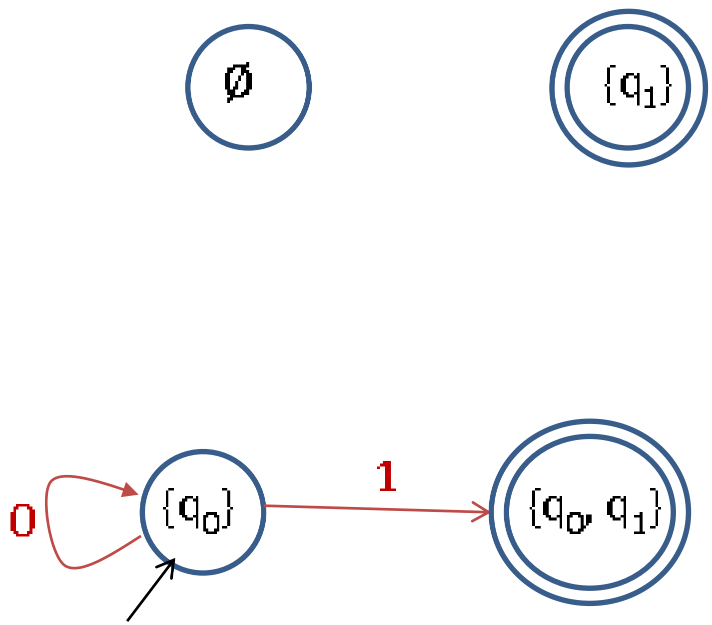




Example: NFA $\Rightarrow$ DFA

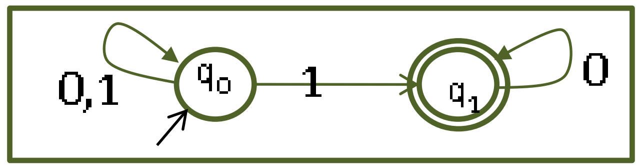
Deterministic Automata - transitions from $\{q_0\}$:

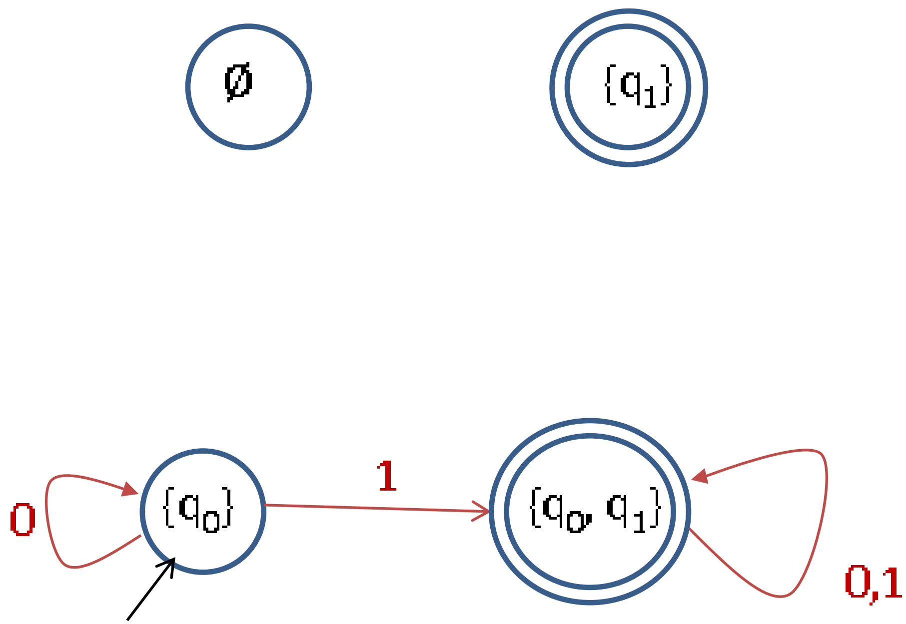




Example: NFA $\Rightarrow$ DFA

Deterministic Automata:





Dealing with ε-Transitions

Given NFA with ε-transitions, we create an equivalent NFA with no ε-transitions.
For any state R of M, define $E(R)$ to be the collection of states reachable from R by ε transitions only.

$E(R) = \{q \in Q | q$ can be reached from some $r \in R$ by 0 or more ε transitions$\}$

Define **transition function**:

$\delta'(R, a) = \{q \in Q| \; q \in E(\delta(r, a))$ for some $r \in R\}$

Change **start state** to

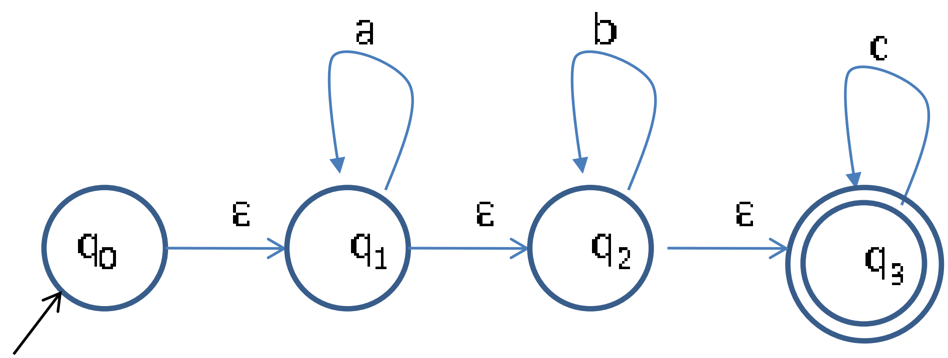
$$q'_0 = E(\{q_0\})$$

♣

Example.

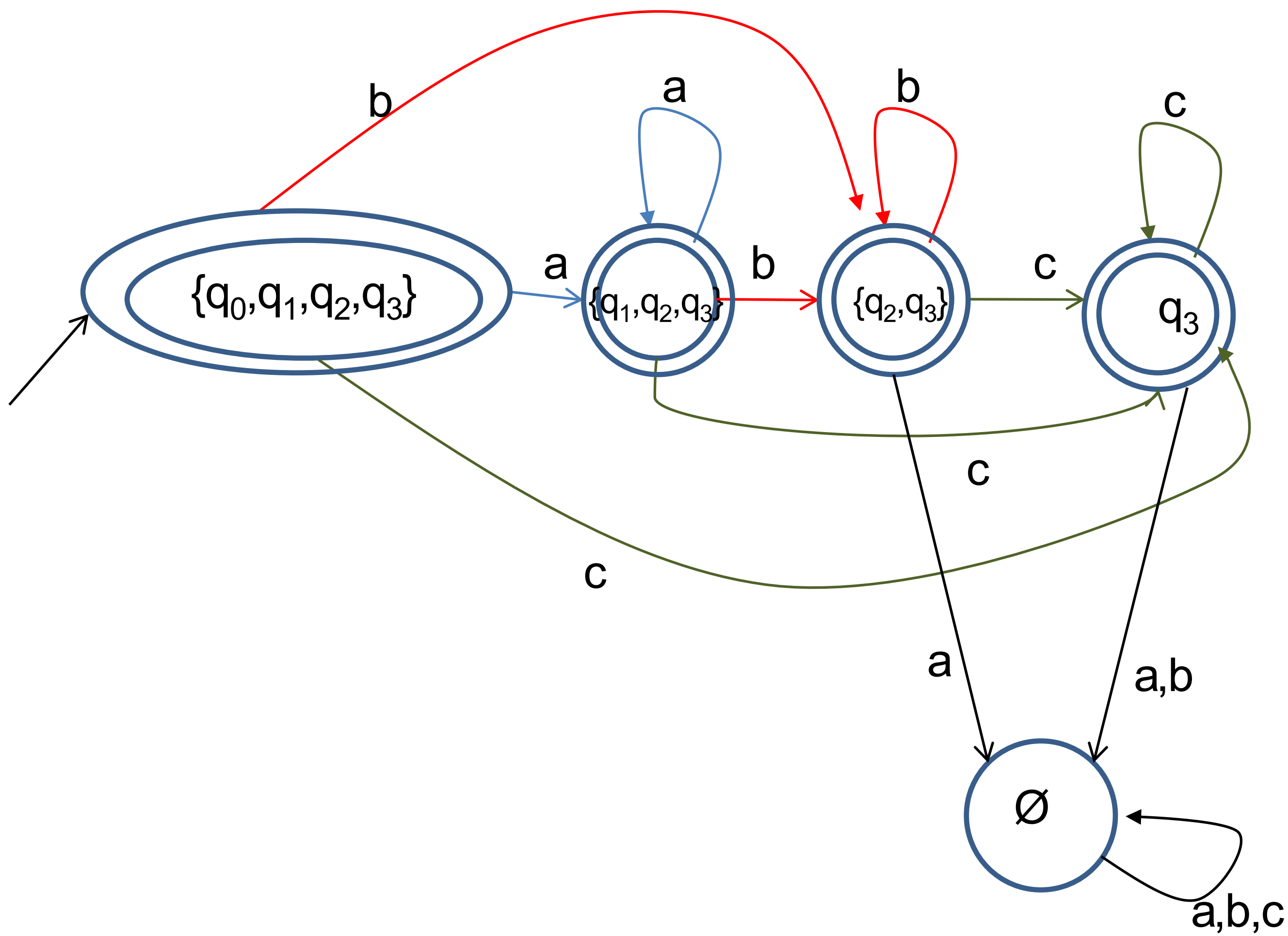
Example: Removing ε-Transitions

Non-Deterministic Automata with ε-tansitions



Example: Removing ε-Transitions

Non-Deterministic Automata without ε-tansitions



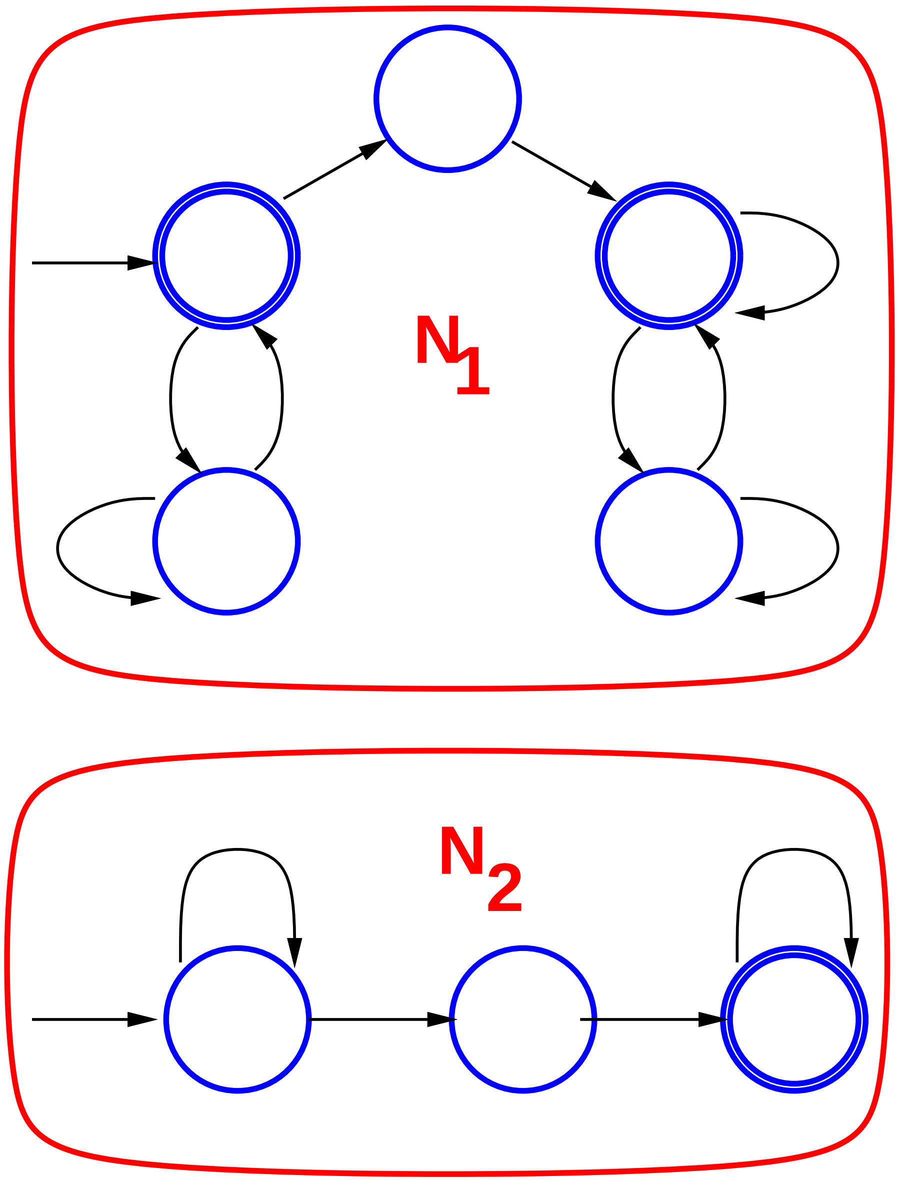
Regular Languages, Revisited

By definition, a language is regular if it is accepted by some DFA.

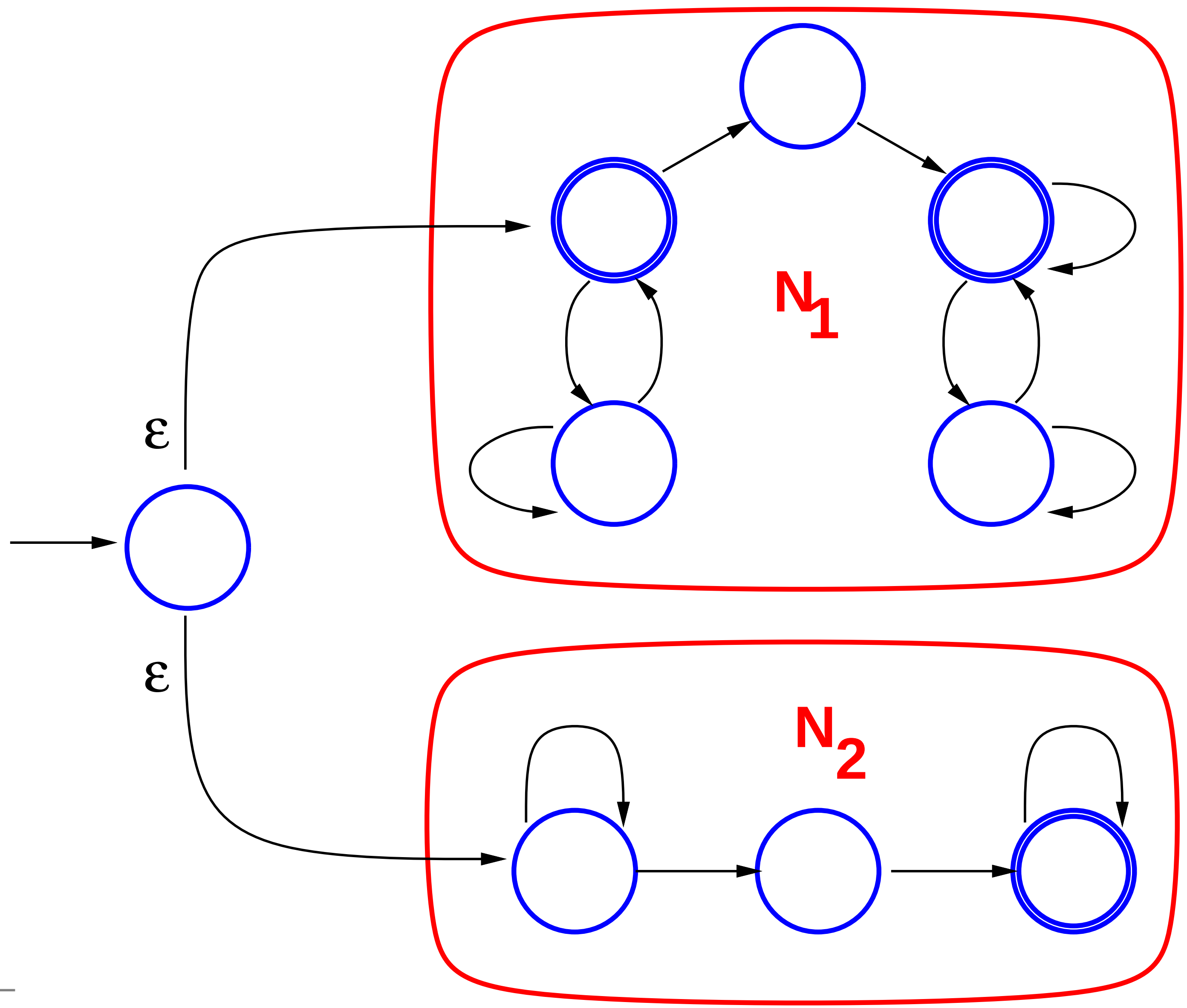
Corollary: A language is regular if and only if it is accepted by some NFA.

This is an alternative way of characterizing regular languages.
We will now use the equivalence to show that regular languages are closed under the regular operations (union, concatenation, star).

Closure Under Union (alternative proof)



Regular Languages Closed Under Union



Regular Languages Closed Under Union

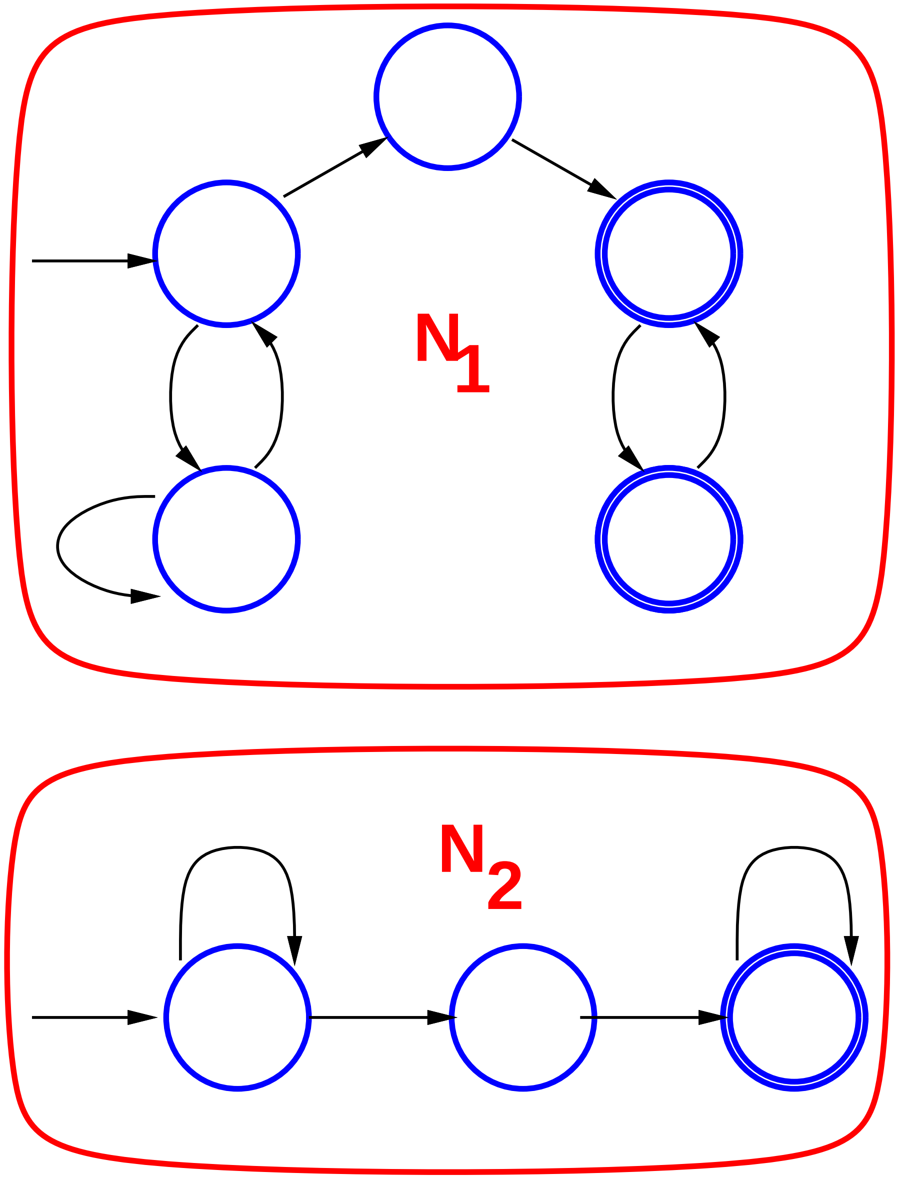
Suppose

- $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ accept L_1, and
- $N_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ accept L_2.

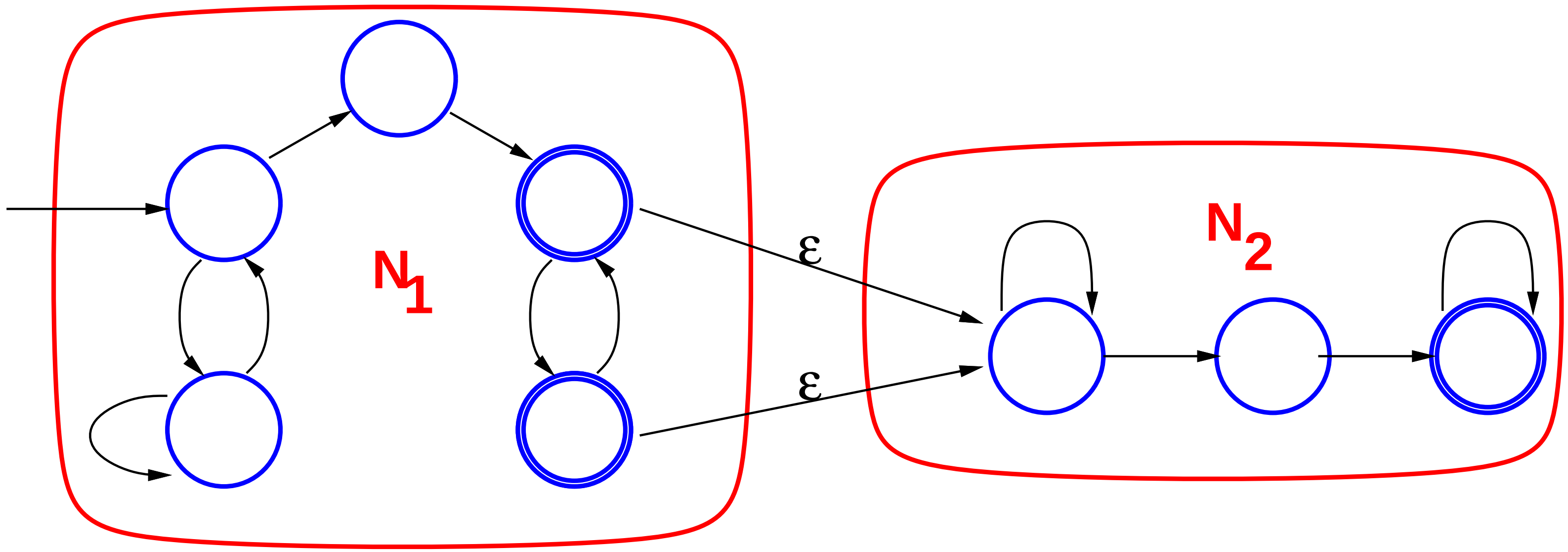
Define $N = (Q, \Sigma, \delta, q_0, F)$:

- $Q = \{q_0\} \cup Q_1 \cup Q_2$ (we assume $Q_1 \cap Q_2 = \emptyset$)
- Σ is the same, q_0 is the start state
- $F = F_1 \cup F_2$
- $$\delta'(q, a) = \begin{cases} \delta_1(q, a) & q \in Q_1 \\ \delta_2(q, a) & q \in Q_2 \\ \{q_1, q_2\} & q = q_0 \text{ and } a = \varepsilon \\ \emptyset & q = q_0 \text{ and } a \neq \varepsilon \end{cases}$$

Regular Languages Closed Under Concatenation



Regular LanguagesClosed Under Concatenation



Remark: Final states are exactly those of N_2.

Regular Languages Closed Under Concatenation

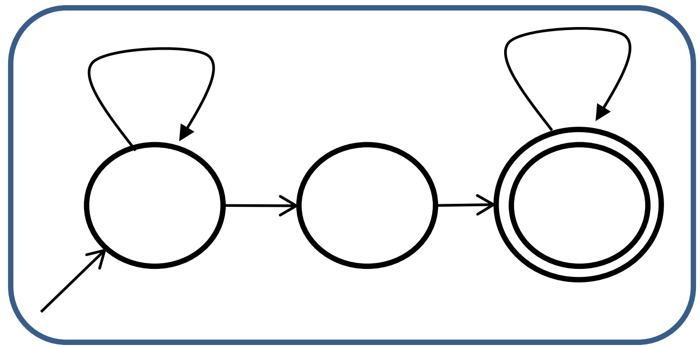
Suppose

- $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ accept L_1, and
- $N_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ accept L_2.

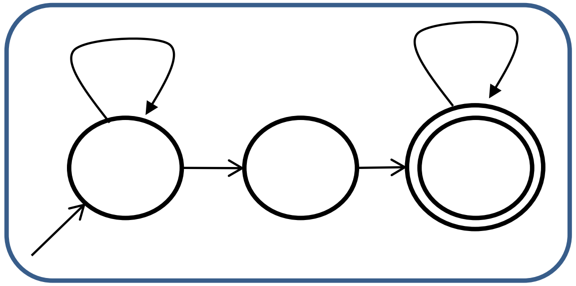
Define $N = (Q, \Sigma, \delta, q_1, F_2)$:

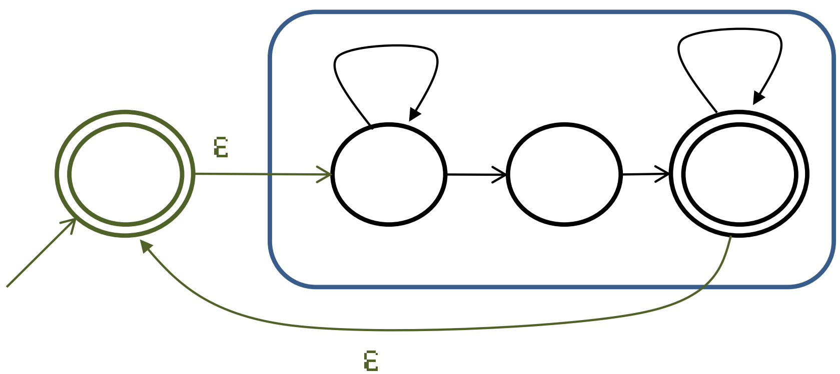
- $Q = Q_1 \cup Q_2$
- q_1 is the start state of N
- F_2 is the set of accept states of N
- $$\delta'(q,a) = \begin{cases} \delta_1(q,a) & q \in Q_1 \text{ and } q \notin F_1 \\ \delta_1(q,a) & q \in Q_1 \text{ and } a \neq \varepsilon \\ \delta_1(q,a) \cup \{q_2\} & q \in F_1 \text{ and } a = \varepsilon \\ \delta_2(q,a) & q \in Q_2 \end{cases}$$

Regular Languages Closed Under Star



Regular Languages Closed Under Star





Regular Languages Closed Under Star

Suppose $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ accepts L_1.
Define $N = (Q, \Sigma, \delta, q_0, F)$:

- $Q = \{q_0\} \cup Q_1$
- q_0 is the new start state.
- $F = \{q_0\} \cup F_1$
- $$\delta'(q,a) = \begin{cases} \delta_1(q,a) & q \in Q_1 \text{ and } q \notin F_1 \\ \delta_1(q,a) & q \in F_1 \text{ and } a \neq \varepsilon \\ \delta_1(q,\varepsilon) \cup \{q_0\} & q \in F_1 \text{ and } a = \varepsilon \\ \{q_1\} & q = q_0 \text{ and } a = \varepsilon \\ \emptyset & q = q_0 \text{ and } a \neq \varepsilon \end{cases}$$
- Example

Summary

- Regular languages are closed under
  - union
  - concatenation
  - star
- Non-deterministic finite automata
  - are equivalent to deterministic finite automata
  - but much easier to use in some proofs and constructions.

Regular Expressions

A notation for building up languages by describing them as expressions, *e.g.* $(0 \cup 1)0^*$.

- 0 and 1 are shorthand for $\{0\}$ and $\{1\}$
- so $(0 \cup 1) = \{0, 1\}$.
- 0^* is shorthand for $\{0\}^*$.
- concatenation, like multiplication, is implicit, so 0^*10^* is shorthand for the set of all strings over $\Sigma = \{0, 1\}$ having exactly a single 1.

Q.: What does $(0 \cup 1)0^*$ stand for?

Remark: Regular expressions are often used in text editors or shell scripts.

More Examples

Let Σ be an alphabet.

- The regular expression Σ is the language of one-symbol strings.
- Σ^* is all strings.
- $\Sigma^* 1$ all strings ending in 1.
- $0\Sigma^* \cup \Sigma^* 1$ strings starting with 0 or ending in 1.

Just like in arithmetic, operations have precedence:

- star first
- concatenation next
- union last
- parentheses used to change default order

Regular Expressions – Formal Definition

Syntax: R is a regular expression over Σ, if R is of form

- a for some $a \in \Sigma$
- ε
- $\emptyset$
- $(R_1 \cup R_2)$ for regular expressions R_1 and R_2
- $(R_1 \circ R_2)$ for regular expressions R_1 and R_2
- (R_1^*) for regular expression R_1

Regular Expressions – Formal Definition

Let $L(R)$ be the language denoted by regular expression R.

R	$L(R)$
a	$\{a\}$
ε	$\{\varepsilon\}$
$\emptyset$	$\emptyset$
$(R_1 \cup R_2)$	$L(R_1) \cup L(R_2)$
$(R_1 \circ R_2)$	$L(R_1) \circ L(R_2)$
$(R_1)^*$	$L(R_1)^*$

Q.: What's the difference between $\emptyset$ and ε?

Q.: Isn't this definition circular?

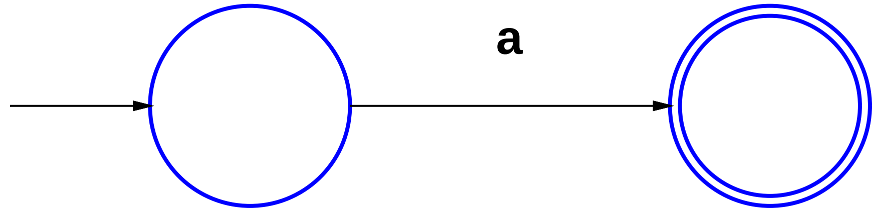
Remarkable Fact

Thm.: A language, L, is described by a regular expression, R, if and only if L is regular.

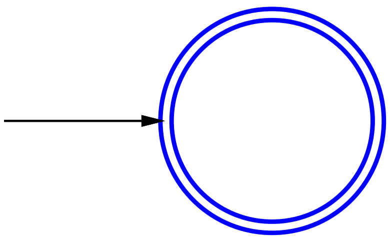
$\Longrightarrow$ construct an NFA accepting R.

$\Longleftarrow$ Given a regular language, L, construct an equivalent regular expression.

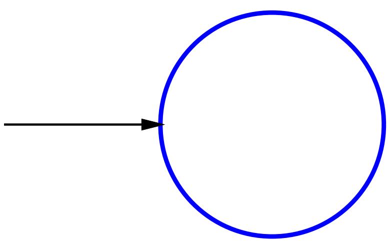
Given R, Build NFA Accepting It ($\Longrightarrow$)



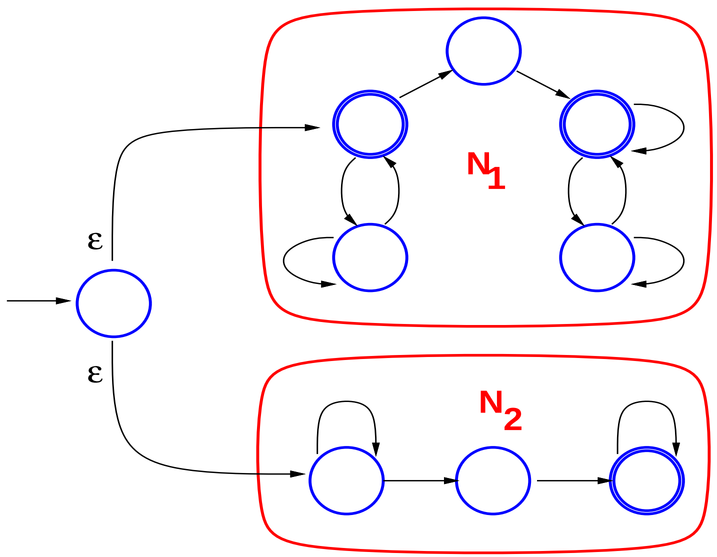
1. $R = a$, for some $a \in \Sigma$
2. $R = \varepsilon$



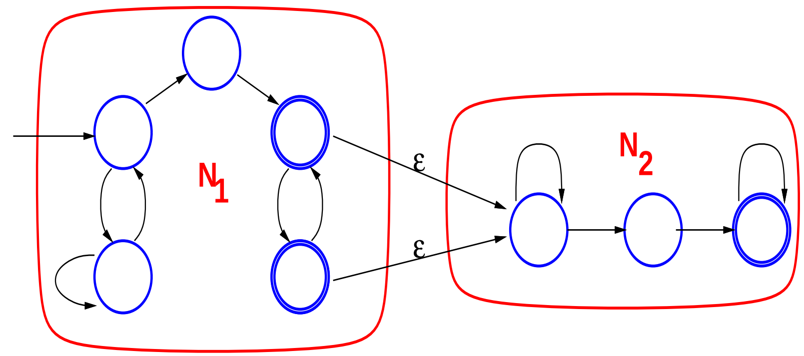
3. $R = \emptyset$



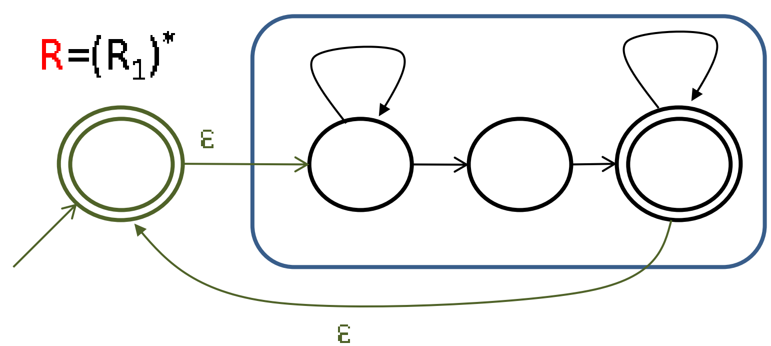
Given R, Build NFA Accepting It ($\Longrightarrow$)



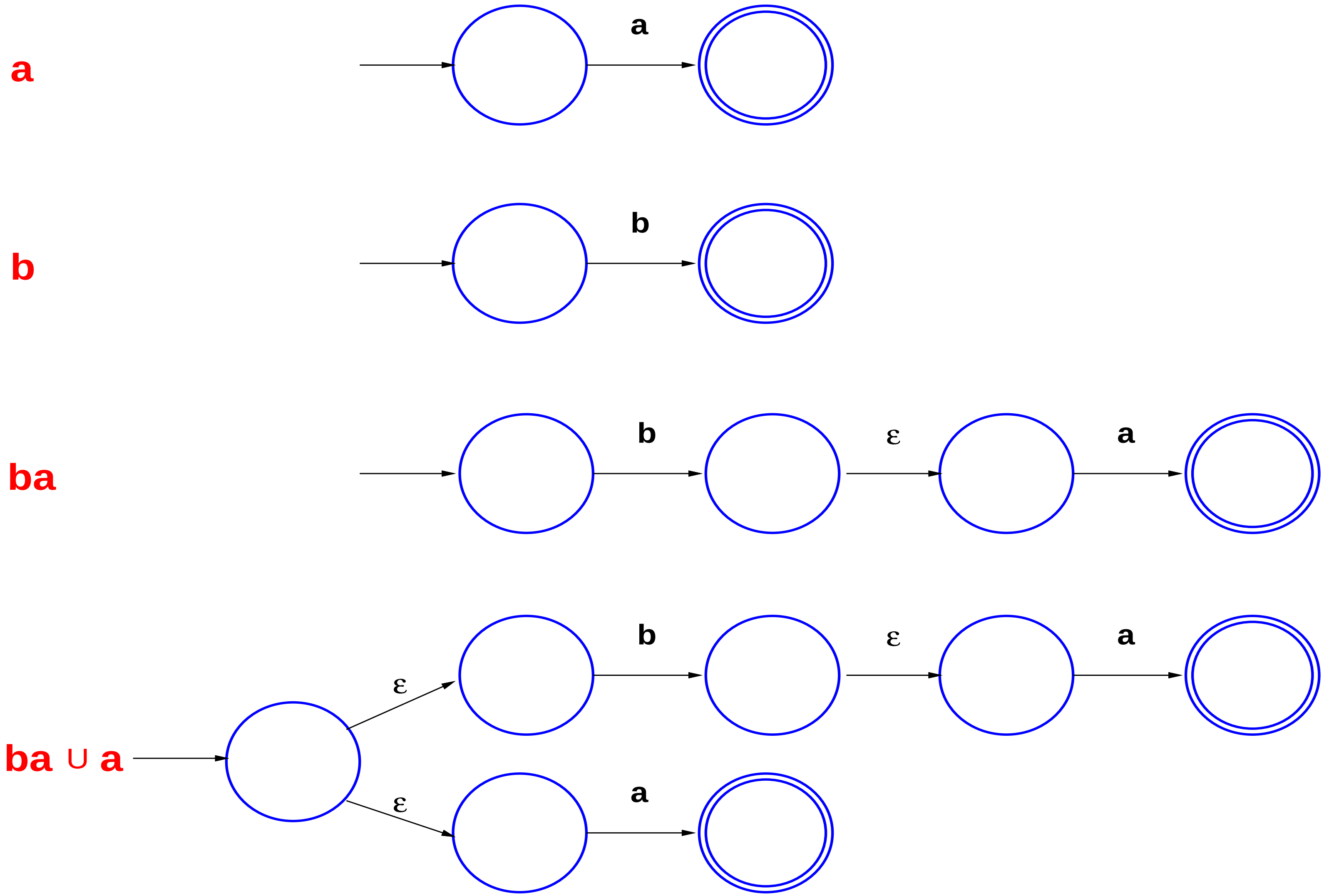
$$R = (R_1 \cup R_2)$$



$$R = (R_1 \circ R_2)$$



Example



Regular Expression from an NFA ($\Longleftarrow$)

We now define generalized non-deterministic finite automata (GNFA).

An NFA:

- Each transition labeled with a symbol or ε,
- reads zero or one symbols,
- takes matching transition, if any.

A GNFA:

- Each transition labeled with a regular expression,
- reads zero or more symbols,
- takes transition whose regular expression matches string, if any.

GNFAs are natural generalization of NFAs.

GNFA – Formal Definition

A generalized deterministic finite automaton (GNFA) is $(Q, \Sigma, \delta, q_s, q_a)$, where

- Q is a finite set of states,
- Σ is the alphabet,
- $\delta : (Q - \{q_a\}) \times (Q - \{q_s\}) \to \mathcal{R}$ is the transition function.
- $q_s \in Q$ is the start state, and
- $q_a \in Q$ is the unique accept state.

GNFA – Model of Computation

A GNFA accepts a string $w \in \Sigma^*$ if there exists a *parsing* of w, $w = w_1 w_2 \cdots w_k$, where each $w_i \in \Sigma^*$, and there exists a sequence of states $q_0, \ldots, q_k$ such that

- $q_0 = q_s$, the start state,
- $q_k = q_a$, the accept state, and
- for each i, $w_i \in L(R_i)$, where $R_i = \delta(q_{i-1}, q_i)$.
  - (namely w_i is an element of the language described by the regular expression R_i.)

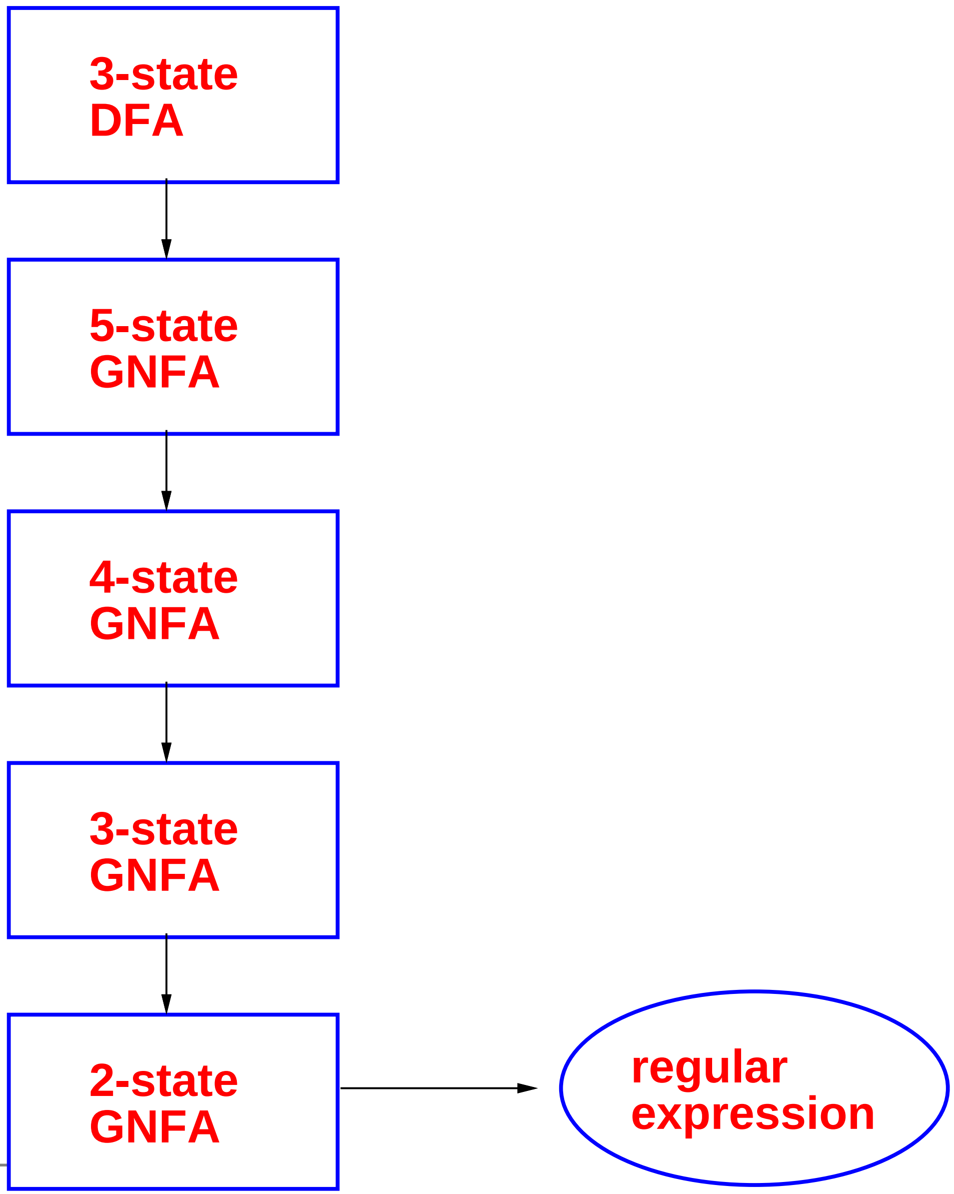
The Transformation: DFA $\rightarrow$ Regular Expression

Strategy – sequence of equivalent transformations

- given a k-state DFA
- transform into $(k+2)$-state GNFA (how?)
  - while GNFA has more than 2 states, transform it into equivalent GNFA with one fewer state
- eventually reach 2-state GNFA (states are just start and accept).
- label on single transition is the desired regular expression.

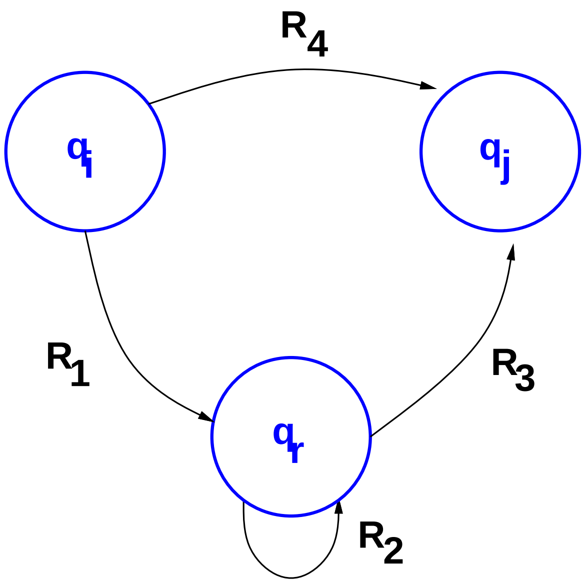
♠

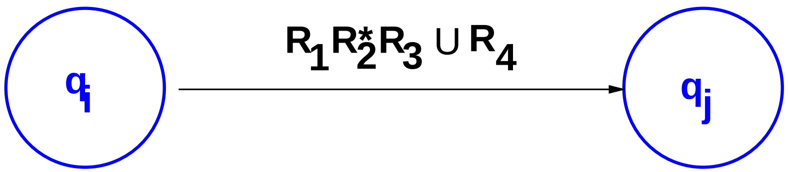
Converting Strategy ($\Longleftarrow$)



Removing One State

We remove one state q_r, and then repair the machine by altering regular expression of other transitions.





The StateReduce Algorithm

Given a $k > 2$-state GNFA G, convert it to an equivalent GNFA G'.

- Select any q_r distinct from q_s and q_a.
- Let $Q' = Q - \{q_r\}$.
- For any $q_i \in Q' - \{q_a\}$ and $q_j \in Q' - \{q_s\}$, let
  - $R_1 = \delta(q_i, q_r)$, $R_2 = \delta(q_r, q_r)$,
  - $R_3 = \delta(q_r, q_j)$, and $R_4 = \delta(q_i, q_j)$.
- Define $\delta'(q_i, q_j) = (R_1)(R_2)^*(R_3) \cup (R_4)$.
- Return the resulting $(k-1)$-state GNFA.

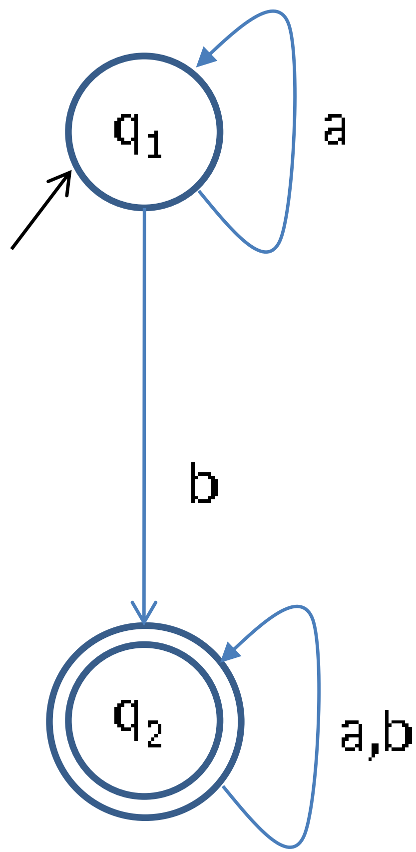
The Convert Algorithm

We define the recursive procedure Convert$(\cdot)$:

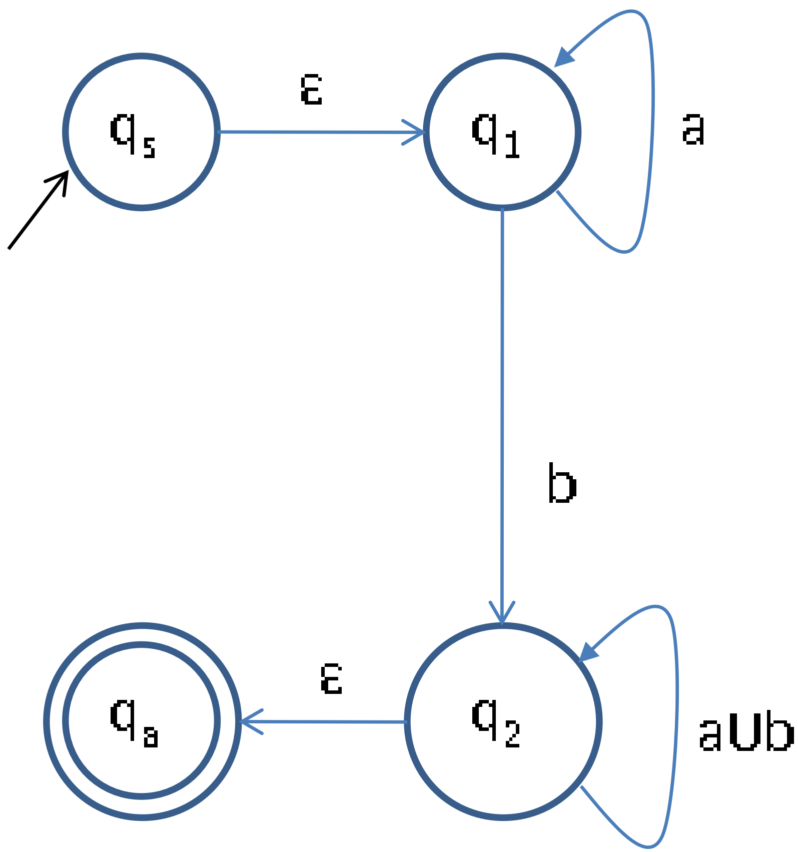
Given GNFA G.

- Let k be the number of states of G.
- If $k = 2$, return the regular expression labeling the only arrow of G.
- Otherwise, return Convert(StateReduce(G)).

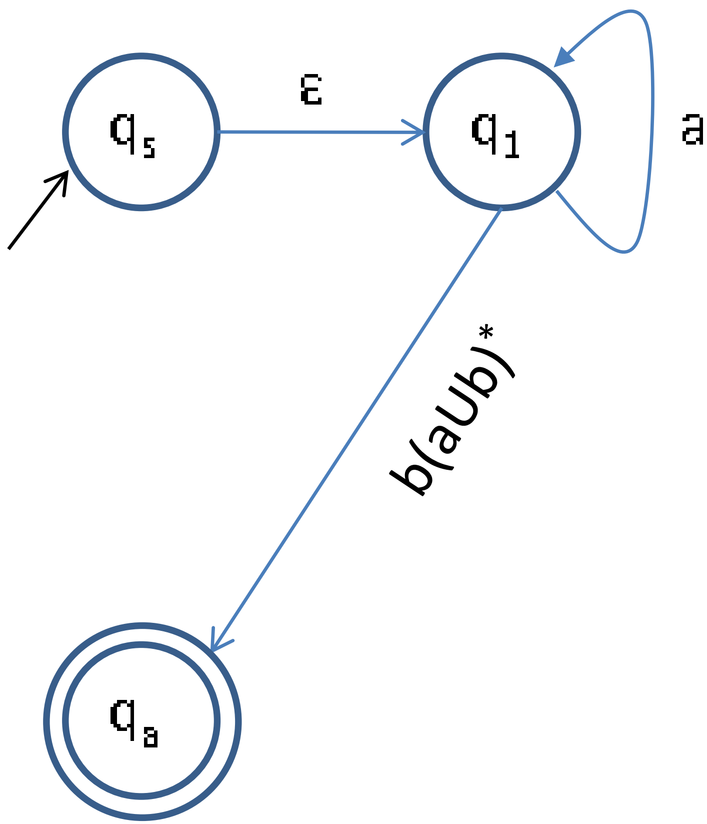
Conversion - Example



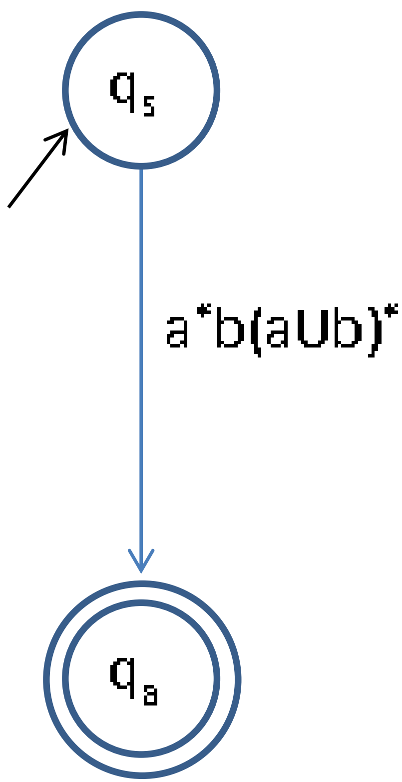
Conversion - Example



Conversion - Example



Conversion - Example



Correctness Proof of Construction

Theorem: G and Convert(G) accept the same language.

Proof: By induction on number of states of G

Basis. $k = 2$: Immediate by the definition of GFNA.

Induction Step: Assume claim for $(k-1)$-state GNFA, where $k > 2$, prove for k-state GNFA.

Let $G' =$ StateReduce(G) (note that G' has $k-1$ states). We prove that $L(G) = L(G')$ (i.e., G and G' accept the same language).

G and G' accept the same language

Two steps:

- If G accepts the string w, then so does G'.
- If G' accepts the string w, then so does G.

Therefore, $L(G) = L(G')$.

Step One

Claim: If G accepts w, then so does G':

- If G accepts w, then there exists a "path of states" $q_s, q_1, q_2, \ldots, q_a$ traversed by G on w.
- If q_r does not appear on path, then G' accepts w because the the new regular expression on each edge of G' contains the old regular expression in the "union part".
- If q_r does appear, consider the regular expression corresponding to $\ldots q_i, q_r, \ldots, q_r, q_j \ldots$.
The new regular expression $(R_{i,r})(R_{r,r})^*(R_{r,j})$ linking q_i and q_j encompasses any such string.
- In both cases, the claim holds.

Step Two

Claim: If G' accepts w, then so does G.

Proof: Each transition from q_i to q_j in G' corresponds to a transition in G, either directly or through q_r. Thus if G' accepts w, then so does G.

Conclusion

- We proved $L(G) = L(G')$.
- Hence, G and (the regular expression) Convert(G) accept the same language.
- Thus, we proved the *remarkable claim*:
  A language, L, is described by a regular expression, R, if and only if L is regular. ♣