Computational Models

Introduction to the Theory of Computing

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Spring Semester, 2012. Mondays, 13–16 and Wednesdays, 10–13.

Course site: http://tau-cm2012.wikidot.com/

Site (containing a forum) is our sole mean of disseminating information.

AdministraTrivia

Course Requirements:

- 6 problem sets
- Readable, concise, correct answers expected.
- Late submission will not be accepted.
- Solving problems independently is highly recommended.
- Midterm, covering first 5 lectures (up to computability).
- Midterm consists of multiple choice problems.
- Midterm tentatively scheduled to Friday, May 4, 2012.
- Final exam, covering all course material.

AdministraTrivia II

- You must pass the final exam to get a passing course grade.
- Grade: 0.75*Exam + 0.15*Midterm + 0.10*HW
- Prerequisites (formally): Extended introduction to computer science
  - But most importantly is “mathematical maturity”.
  - Students from other disciplines with mathematical background encouraged to contact the instructor.

- Textbook:
  - Michael Sipser, *Introduction to the theory of computation*, 1st or 2nd edition.

Why Study Theory?

- Basic Computer Science Issues
  - What is a computation?
  - Are computers omnipotent?
  - What are the fundamental capabilities and limitations of computers?
- Pragmatic Reasons
  - Avoid intractable or impossible problems.
  - Apply efficient algorithms when possible.
  - Learn to tell the difference.

Course Topics

- Automata Theory: Basic model of computation.
  - Re-invented in many other disciplines.
- Computability Theory: What can computers do?
  - True impossibility results.
- Complexity Theory: What makes some problems computationally hard and others easy?
- Coping with intractability.

Automata Theory - Simple Models

- Finite automata.
  - Related to controllers and hardware design.
  - Useful in text processing and finding patterns in strings.
  - Probabilistic (Markov) versions useful in modeling various natural phenomena (*e.g.* speech recognition).
- Push down automata.
  - Tightly related to a family of languages known as context free languages.
  - Play important role in compilers, design of programming languages, and studies of natural languages.

Computability Theory

In the first half of the 20th century, mathematicians such as Kurt Göedel, Alan Turing, and Alonzo Church discovered that some fundamental problems cannot be solved by computers.

- Proof verification of statements can be automated.
- It is natural to expect that determining validity can also be done by a computer.
- Theorem: A computer cannot determine if mathematical statement true or false.
- Results needed theoretical models for computers.
- These theoretical models helped lead to the construction of real computers.

Halting Problem

Can we decide if a program would halt on a given input.

- Proof idea Show a paradox using such a program HALT
- Simple Paradox:
- The statement below is False
- The statement above is True
- HALT(X) halts if program X does not halt on input X, otherwise does not halt.
- Consider construction of the type: HALT(HALT)

Some Other Undecidable Problems

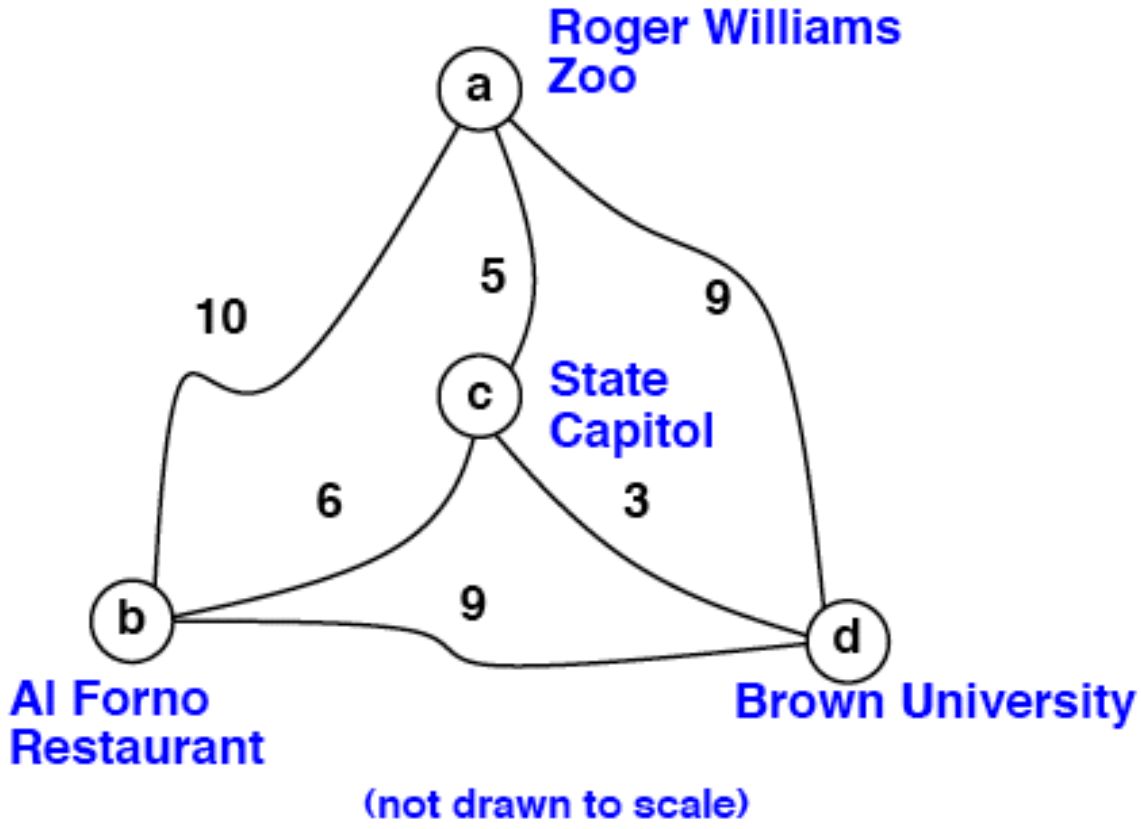
- Does a program run forever?
- Is a program correct?
- Are two programs equivalent?
- Is a program optimal (time wise – is it the fastest program solving a specific problem)?
- Does a polynomial equation with one or more variables and integer coefficients (*e.g.* $5x + 15y + 19xyz^2 = 12$) have an integer solution (Hilbert's 10th problem).
- Given a string, x, how compressible is it?

Complexity Theory

Key notion: tractable vs. intractable problems.

- A problem is a general computational question:
  - description of parameters
  - description of solution
- An algorithm is a step-by-step procedure
  - a recipe
  - a computer program
  - a mathematical object
- We want the most efficient algorithms
  - fastest (usually)
  - most economical with memory (sometimes)
  - expressed as a function of problem size

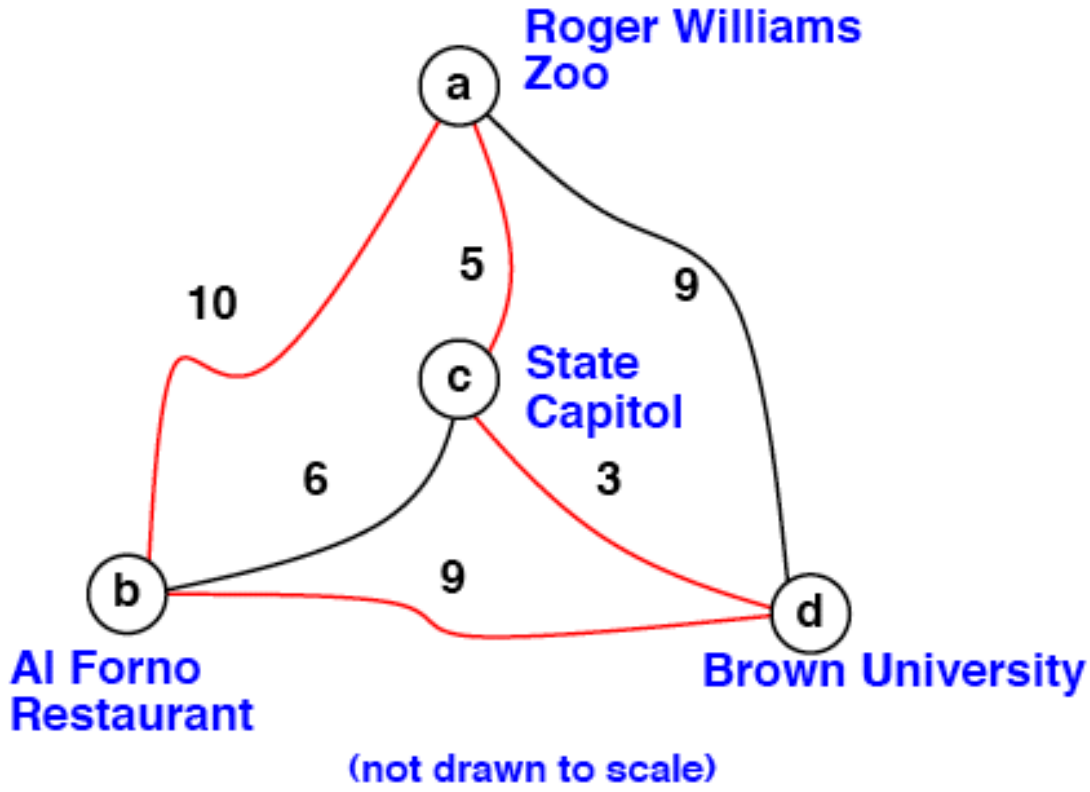
Example: Traveling Salesman Problem



Input:

- set of cities
- set of inter-city distances

Example: Traveling Salesman Problem



Goal:

- want the shortest tour through the cities
- example: `a`, `b`, `d`, `c`, `a` has length 27.

Problem Size

- What is an appropriate measure of problem size?
  - $\mathtt{m}$ nodes?
  - $\mathtt{m}(\mathtt{m}+1)/2$ distances?
- Use an encoding of the problem
  - alphabet of symbols
  - strings: a/b/c/d//10/5/9//6/9//3.
- Measures
  - Problem Size: length of encoding (above: 23 ascii characters).
  - Time Complexity: how long an algorithm runs, as function of problem size?

Time Complexity - What is tractable?

- We say that a function $f(n)$ is $O(g(n))$ if there is a constant c such that for large enough n, $|f(n)| \leq c \cdot |g(n)|$.
- A polynomial-time algorithm is one whose time complexity is $O(p(n))$ for some polynomial $p(n)$, where n denotes the length of the input.
- An exponential-time algorithm is one whose time complexity cannot be bounded by a polynomial (e.g., $n^{\log n}$).

Tractability – Basic distinction:

- Polynomial time = tractable.
- Exponential time = intractable.

	10	20	30	40	50	60
n	.00001 second	.00002 second	.00003 second	.00004 second	.00005 second	.00006 second
n^2	.00001 second	.00004 second	.00009 second	.00016 second	.00025 second	.00036 second
n^3	.00001 second	.00008 second	.027 second	.064 second	.125 second	.216 second
n^5	.1 second	3.2 seconds	24.3 seconds	1.7 minute	5.2 minutes	13.0 minutes
2^n	.001 second	1.0 second	17.9 minutes	12.7 days	35.7 years	366 centuries
3^n	.059 second	58 minutes	6.5 years	3855 centuries	$2 \cdot 10^8$ centuries	$1.3 \cdot 10^{13}$ centuries

Effect of Speed-Ups

Let's wait for faster hardware! Consider maximum problem size you can solve in an hour.

	present	100 times faster	1000 times faster
n	N_1	$100N_1$	$1000N_1$
n^2	N_2	$10N_2$	$31.6N_2$
n^3	N_3	$4.64N_3$	$10N_3$
n^5	N_4	$2.5N_4$	$3.98N_4$
2^n	N_5	$N_5 + 6.64$	$N_5 + 9.97$
3^n	N_6	$N_6 + 4.19$	$N_6 + 6.29$

NP-Completeness / NP-Hardness

- Set of interesting optimization problems
- Exhaustive checking is exponential.
- all known algorithms are exponential
- actual complexity not proven
- Believed to require exponential time

Coping with NP-completeness

Intractability isn't everything.

- Find an approximate solution (is a solution within 10% of optimum good enough, ma'am?).
- Use randomization.
- Fixed parameter algorithms may be applicable.
- Heuristics can also help.

- Approximation, randomization, etc. are among the hottest areas in complexity theory and algorithmic research today.

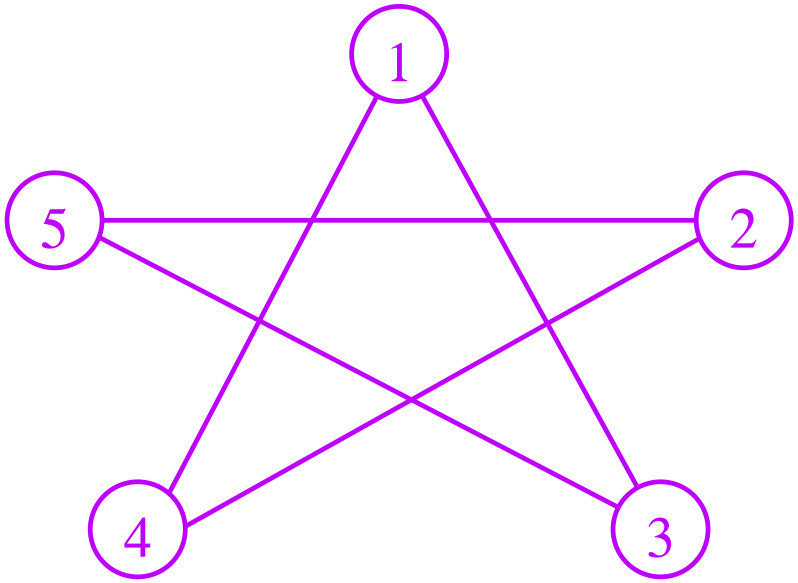
Next Subject

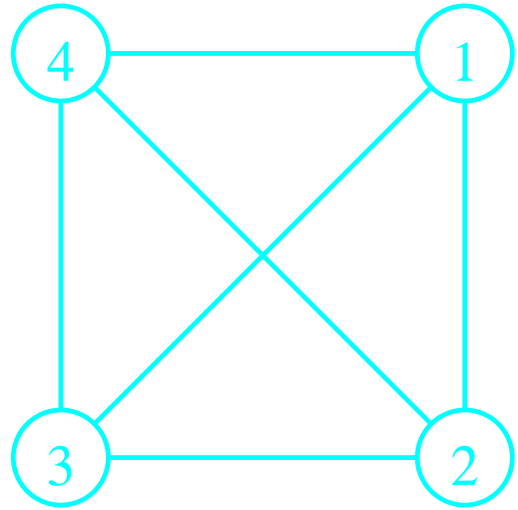
A Very Short Math Review

- Graphs
- Strings and languages
- Mathematical proofs
- Mathematical notations (sets, sequences, . . .) $\surd$
- Functions and predicates $\surd$

$\surd$ = will be done in recitation.

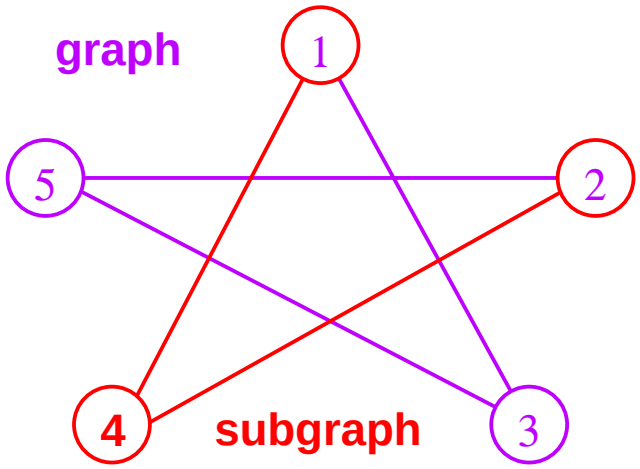
Graphs

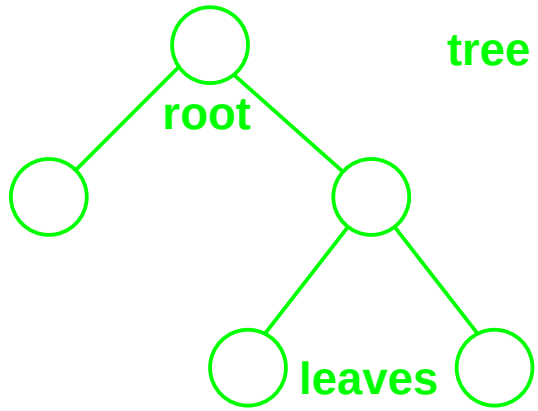


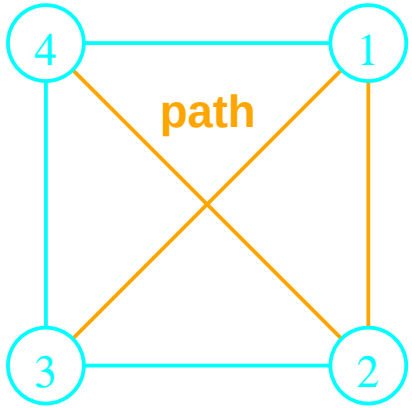


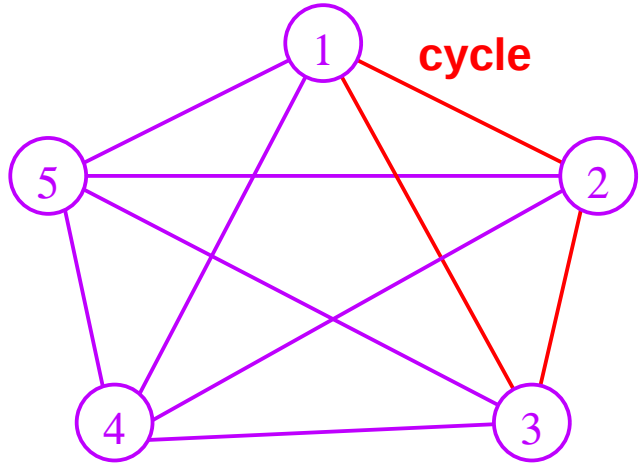
- $G = (V, E)$, where
- V is set of nodes or vertices, and
- E is set of edges
- degree of a vertex is number of edges

Graphs

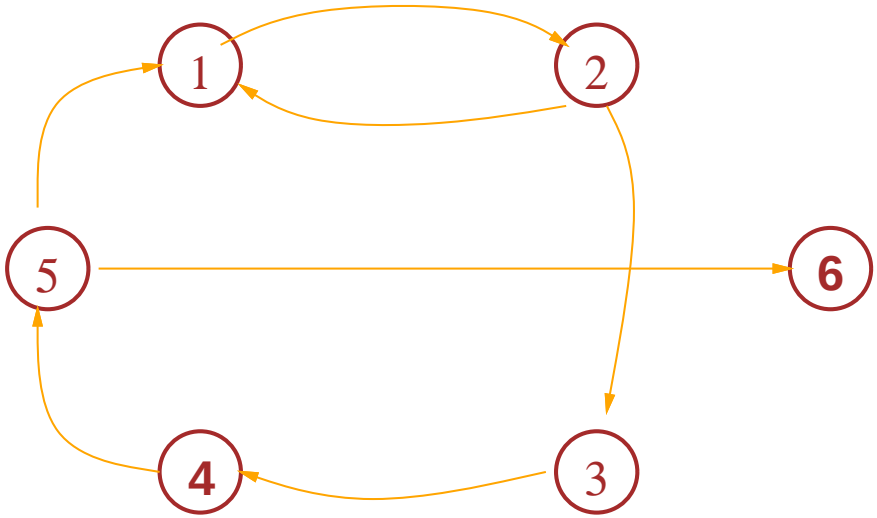

graph
1
5
2
4
subgraph
3

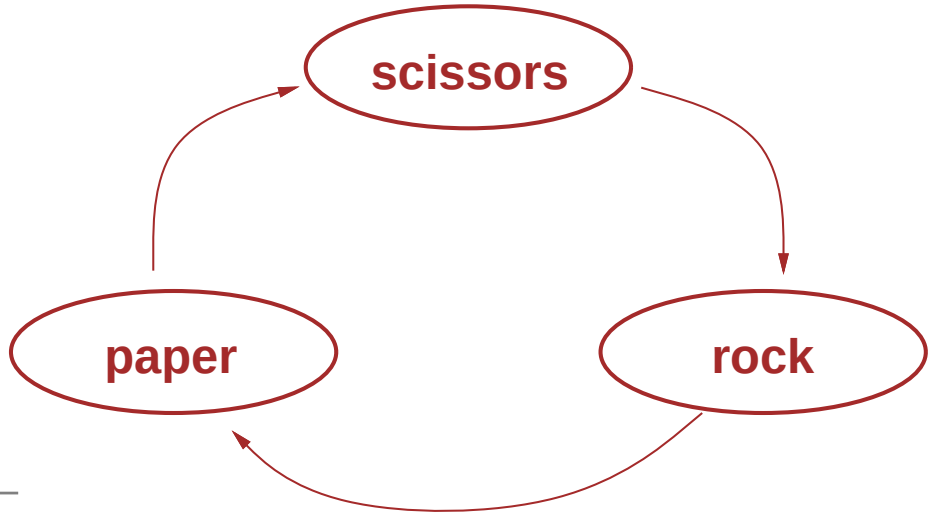

tree
root
leaves


4
1
path
3
2


1
cycle
5
2
4
3

Directed Graphs





Directed Graph and its Adjacency Matrix

Which directed graph is represented by the following 6-by-6 matrix?

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Strings and Languages

- an alphabet is a finite set of symbols
- a string over an alphabet is a finite sequence of symbols from that alphabet.
- the length of a string is the number of symbols
- the empty string ε
- reverse: $abcd$ reversed is $dcba$.
- substring: xyz in $xyzzy$.
- concatenation of xyz and zy is $xyzzy$.
- x^k is $x \cdots x$, k times.
- a language L is a set of strings.

Proofs

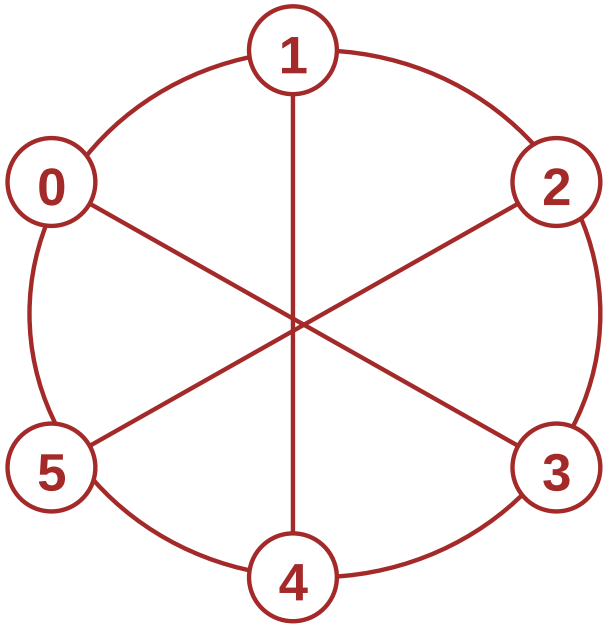
We will use the following basic kinds of proofs.

- by construction
- by contradiction
- by induction
- by reduction
- we will often mix them.

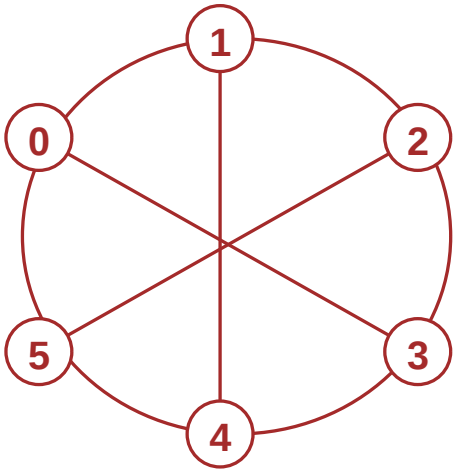
Proof by Construction

A graph is k-regular if every node has degree k.

Theorem: For every even $n > 2$, there exists a 3-regular graph with n nodes.



Proof by Construction



Proof: Construct $G = (V, E)$, where $V = \{0, 1, \ldots, n-1\}$ and

$$\begin{aligned} E &= \{\{i, i+1\} \mid \textit{for } 0 \leq i \leq n-2\} \cup \{n-1, 0\} \\ &\quad \cup \{\{i, i+n/2\} \mid \textit{for } 0 \leq i \leq n/2-1\}\,. \end{aligned}$$

Note: A picture is helpful, but it is *not* a proof!

Proof by Contradiction

Theorem: $\sqrt{2}$ is irrational.

Proof: Suppose not. Then $\sqrt{2} = \frac{m}{n}$, where m and n are relatively prime.

$$\begin{aligned} n\sqrt{2} &= m \\ 2n^2 &= m^2 \end{aligned}$$

Proof by Contradiction (cont.)

So m^2 is even, and so is $m = 2k$.

$$\begin{aligned} 2n^2 &= (2k)^2 \\ &= 4k^2 \\ n^2 &= 2k^2 \end{aligned}$$

Thus n^2 is even, and so is n.

Therefore both m and n are even, and not relatively prime!
Reductio ad absurdam.

Proof by Induction

Prove properties of elements of an infinite set.

$$\mathcal{N} = \{1, 2, 3, \ldots\}$$

To prove that $\wp$ holds for each element, show:

- *base step:* show that $\wp(1)$ is true.
- induction step: show that if $\wp(i)$ is true (the induction hypothesis), then so is $\wp(i+1)$.

Induction Example

Theorem: All cows are the same color.
Base step: A single-cow set is definitely the same color.
Induction Step: Assume all sets of i cows are the same color. Divide the set $\{1, \ldots, i+1\}$ into $U = \{1, \ldots, i\}$, and $V = \{2, \ldots, i+1\}$.

All cows in U are the same color by the induction hypothesis.
All cows in V are the same color by the induction hypothesis.
All cows in $U \cap V$ are the same color by the induction hypothesis.

Induction Example (cont.)

Ergo, all cows are the same color.

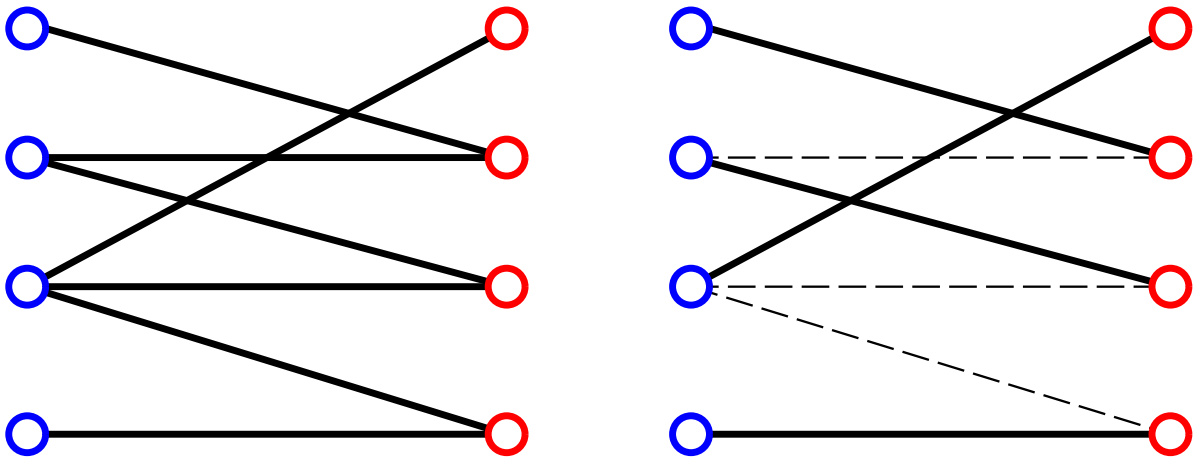
Quod Erat Demonstrandum (QED).

(cows' images courtesy of www.crawforddirect.com/ cows.htm)

Proof by Reduction

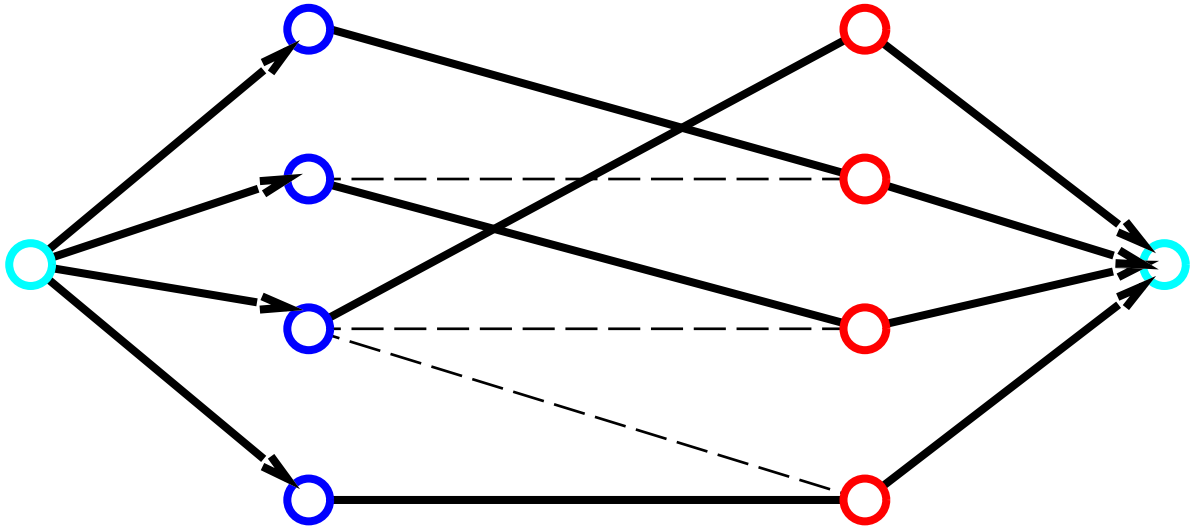
We can sometime solve problem A by reducing it to problem B, whose solution we already know.

Example: Maximal matching in bipartite graphs:



Proof by Reduction

Reducing bipartite matching to MAX FLOW:



Reduction: Put capacity 1 on each edge.

Maximum flow corresponds to maximum matching. So if we have an algorithm that produces max flow, we can easily derive a maximum bipartite matching from it.