

# Computational Models - Lecture 8<sup>1</sup>

## Handout Mode

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<sup>1</sup>Based on slides by Benny Chor, Tel Aviv University, modifying slides by Maurice Herlihy, Brown University.

# Talk Outline

- Mapping Reductions
- Undecidability by Rice Theorem
- Sipser's book, [5.1–5.3](#)

# Section 1

## Mapping Reductions

## Reminder

We have already

- Established Turing Machines as the **gold standard** of computers and computability ...
- seen examples of solvable problems ...
- and saw one problem,  $A_{TM}$ , that is computationally unsolvable.

Today, we look at other computationally unsolvable problems via reductions and introduce the techniques of **mapping reductions**.

# Computable Functions

## Definition 1 (total computable functions)

A TM  $M$  computes a **total** function  $f : \Sigma^* \rightarrow \Sigma^*$ , if when starting with an input  $w$ ,  $M$  halts with (only)  $f(w)$  written on tape.<sup>a</sup>

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<sup>a</sup>The definition naturally extends to functions of **more than one** variable, where some special separator symbol indicates end of one variable and beginning of next.

## Definition 2 (partially computable functions)

A TM  $M$  computes a **partial** function  $f : \Sigma^* \rightarrow (\Sigma^* \cup \perp)$ , if when starting with an input  $w$ :

- if  $f(w)$  is defined (i.e.,  $\neq \perp$ ),  $M$  halts with only  $f(w)$  on tape,
- if  $f(w)$  is undefined,  $M$  **does not halt**.

Computable functions are also called (**total** or **partial**) **recursive** functions.

## Example, 1

### Claim 3

All the “usual” arithmetic functions on integers are computable.

These include addition, subtraction, multiplication, division (quotient and remainder), exponentiation, roots (to a specified precision), modular exponentiation, greatest common divisor.

Even **non-arithmetic** functions, like logarithms and trigonometric functions, can be computed (to a specified precision), using Taylor expansion or other numeric mathematic techniques.

### Exercise 4

Design a TM that on input  $\langle m, n \rangle$ , halts with  $\langle m + n \rangle$  on tape.

## Example, 2

A useful class of functions **modifies TM descriptions**. For example:

### Algorithm 5

On input  $w$ : If  $w = \langle M \rangle$  for some TM, construct  $\langle M' \rangle$ , where

- $L(M') = L(M)$ , but
- $M'$  never tries to move off LHS of tape.

Otherwise, write  $\varepsilon$  and halt.

### Question 6

Is the function defined above total? computable?

# Reducibility

- Finding your way around a new city, reduces to ... obtaining a city map.
- Finding the median in an array, reduces to ... sorting an array
- Computing a maximum matching in a bi-partite graph, reduces to ... computing a maximum flow problem.
- The core idea behind [procedures](#)



## Reducibility, In Our Context

Involves two problems,  $A$  and  $B$ .

**Desired property:** If  $A$  reduces to  $B$ , then any solution of  $B$  can be used to find a solution of  $A$ .

### Remark 7

This property says nothing about solving  $A$  by itself or  $B$  by itself.

but

### Fact 8

If  $A$  reduces to  $B$ , then  $A$  cannot be *harder* than  $B$

- if  $B$  is decidable, so is  $A$ .
- if  $A$  is *undecidable*, then  $B$  is *undecidable*.

We next use reductions and the undecidability of  $A_{TM}$ , to show the undecidability of several problems.

## $H_{TM}$ is undecidable

- $A_{TM} = \{\langle M, w \rangle \mid M \text{ is a TM that accepts } w\}$
- $H_{TM} = \{\langle M, w \rangle \mid M \text{ is a TM and } M \text{ halts on input } w\}$

### Theorem 9

$H_{TM}$  is undecidable.

Proof: Assume, by way of contradiction, that TM  $R$  decides  $H_{TM}$ .

### Algorithm 10 (S)

On input  $\langle M, w \rangle$ ,

- 1 Emulate  $R$  on  $\langle M, w \rangle$ .
- 2 If  $R$  rejects, **reject**.
- 3 If  $R$  accepts (meaning  $M$  halts on  $w$ ), emulate  $M$  on  $w$  until it halts (namely run  $U$  on  $\langle M, w \rangle$ ).
- 4 If  $M$  accepted, **accept**; otherwise **reject**.

TM  $S$  decides  $A_{TM}$ , a contradiction ♣

What we actually did is a **reduction** from  $A_{TM}$  to  $H_{TM}$ .

## EMPTY<sub>TM</sub> is Undecidable

$$\text{EMPTY}_{\text{TM}} = \{\langle M \rangle \mid M \text{ is a TM and } L(M) = \emptyset\}$$

### Theorem 11

EMPTY<sub>TM</sub> is undecidable.

Proof's idea: By contradiction:

- Assume EMPTY<sub>TM</sub> is decidable and let  $R$  be a TM that decides EMPTY<sub>TM</sub>.
- Use  $R$  to construct  $S$ , a TM that decides  $A_{\text{TM}}$ .

### Algorithm 12 ( $S$ – first attempt)

On input  $\langle M, w \rangle$ :

Emulate  $R(\langle M \rangle)$  and **reject** if  $R$  **accepts**.

But what if  $R$  **rejects**?

## EMPTY<sub>TM</sub> is Undecidable, 2

Solution? Modify  $M$

### Definition 13 ( $M_w$ )

Given a TM  $M$  and input  $w$ , define the TM  $M_w$  as follows:

On input  $x$ ,

- 1 if  $x \neq w$ , **reject**.
- 2 if  $x = w$ , run  $M$  on  $w$  and **accept** if  $M$  does.

$M_w$  either

- accepts **just**  $w$  (in case  $M$  accepts  $w$ ), or
- accepts **nothing** (otherwise).

## EMPTY<sub>TM</sub> is Undecidable, 3

### Definition 14

The language of  $M_w$ :

$$L(M_w) := \begin{cases} \{w\}, & \text{M accepts } w, \\ \emptyset, & \text{M does not accept } w \end{cases}$$

### Question 15

Can a TM construct  $M_w$  from  $\langle M, w \rangle$ ?

**Answer:** Easily, because we need only hardwire  $w$ , and add a few extra states to perform the “ $x = w$ ?” test.

## EMPTY<sub>TM</sub> is Undecidable, 4

- $\text{EMPTY}_{\text{TM}} = \{\langle M \rangle \mid M \text{ is a TM and } L(M) = \emptyset\}$
- Assume  $\text{EMPTY}_{\text{TM}}$  is decidable and let  $R$  be a TM that decides  $\text{EMPTY}_{\text{TM}}$ .

### Algorithm 16 (S)

On input  $\langle M, w \rangle$

- 1 Construct  $M_w$  from  $M$  and  $w$ .
- 2 Emulate  $R$  on input  $\langle M_w \rangle$ .
- 3 If  $R$  accepts, **reject**; if  $R$  rejects, **accept**.

### Claim 17

$S$  decides  $A_{\text{TM}}$ .

A **contradiction**.



## REG<sub>TM</sub> is Undecidable

$$\text{REG}_{\text{TM}} = \{ \langle M \rangle \mid M \text{ is a TM and } L(M) \text{ is regular} \}$$

### Theorem 18

REG<sub>TM</sub> is undecidable.

Proof's idea: By contradiction.

- Assume REG<sub>TM</sub> is decidable and let  $R$  be a TM that decides REG<sub>TM</sub>.
- Use  $R$  to construct  $S$  – a TM that decides  $A_{\text{TM}}$ .

### Question 19

But how we construct  $S$ ?

**Intuition:** On input  $\langle M, w \rangle$ , build  $M_w$  that accepts regular language iff  $M$  accepts  $w$ .

## TM $M_w$

- if  $M$  does **not accept**  $w$ , then  $M_w$  **accepts** the (**non-regular**) language  $\{0^n 1^n \mid n \geq 0\}$
- if  $M$  **accepts**  $w$ , then  $M_w$  **accepts** the (**regular**) language  $\Sigma^*$ .

### Algorithm 20 ( $M_w$ )

On input  $x$ ,

- 1 If  $x$  has the form  $0^n 1^n$ , **accept** it.
- 2 Otherwise, emulate  $M$  on input  $w$  and **accept**  $x$  if  $M$  **accepts**  $w$ .

### Claim 21

- 1 If  $M$  does **not accept**  $w$ , then  $M_w$  **accepts** (the language)  $\{0^n 1^n \mid n \geq 0\}$ .
- 2 If  $M$  **accepts**  $w$ , then  $M_w$  **accepts** (the language)  $\Sigma^*$ .
- 3 The function: **on input**  $\langle M, w \rangle$  **output**  $\langle M_w \rangle$ , is **computable**.



**Algorithm 22 ( $S$ )**

On input  $\langle M, w \rangle$ ,

- 1 Construct  $M_w$  from  $M$  and  $w$ .
- 2 Emulate  $R$  on input  $\langle M_w \rangle$ , where  $R$  be a TM that decides  $REG_{TM}$ .
- 3 If  $R$  accepts, **accept**; if  $R$  rejects, **reject**.

**Claim 23**

$S$  decides  $A_{TM}$ .

A **contradiction**



## $EQ_{TM}$ is Undecidable

$$EQ_{TM} = \{ \langle M_1, M_2 \rangle : M_1, M_2 \text{ are TMs and } L(M_1) = L(M_2) \}$$

### Theorem 24

$EQ_{TM}$  is undecidable.

We are getting tired of reducing  $A_{TM}$  to *everything*.  
Let's try instead a reduction from  $EMPTY_{TM}$  to  $EQ_{TM}$ .

Proof's idea:

- $EMPTY_{TM}$  is the problem of testing whether a TM language is empty.
- $EQ_{TM}$  is the problem of testing whether two TM languages are the same.
- If one of these two TM languages happens to be empty, then we are back to  $EMPTY_{TM}$ .
- So  $EMPTY_{TM}$  is a special case of  $EQ_{TM}$ .

The rest is easy.

## $EQ_{TM}$ is Undecidable, 2

- $EQ_{TM} = \{ \langle M_1, M_2 \rangle : M_1, M_2 \text{ are TMs and } L(M_1) = L(M_2) \}$
- Assume  $EQ_{TM}$  is decidable and let  $R$  be a TM deciding  $EQ_{TM}$ .

Proof:

### Algorithm 25 ( $M_{NO}$ )

On input  $x$ , **reject**

### Algorithm 26 ( $S$ )

On input  $\langle M \rangle$ :

Emulate  $R$  on input  $\langle M, M_{NO} \rangle$ .

If  $R$  accepts, **accept**; if  $R$  rejects, **reject**.

### Claim 27

$S$  decides  $EMPTY_{TM}$ .

A **contradiction** ♣

## Bucket of Undecidable Problems

Same techniques prove undecidability of

- Does a TM accept a **decidable** language?
- Does a TM accept a **context-free** language?
- Does a TM accept a **finite** language?
- Does a TM halt on **all inputs**?
- Is there an input string that causes a TM to **traverse all its states**?

## The Busy Beaver



(taken from <http://www.saltine.org/joebeaver1.jpg>)

## The Busy Beaver, 2

We focus on one tape TMs, with  $\Sigma = \{0, 1\}$  and  $\Gamma = \{0, 1, \sqcup\}$ .

### Definition 28 ( $S_n$ and $BB(n)$ )

For  $n \in \mathbb{N}$ , let  $S_n = \{\text{all } n\text{-state TM's that halt on } \varepsilon\}$ .

Let  $BB(n)$  be maximum  $\#$  of steps taken by some  $M \in S_n$  on input  $\varepsilon$ .

- The set  $S_n$  is finite (under standard encoding)
- Every  $M \in S_n$  runs for **finitely** many steps on  $\varepsilon$ .
- $BB(n)$  is a total function from  $\mathbb{N}$  to  $\mathbb{N}$  (in particular,  $BB(n) \in \mathbb{N}$  for every  $n \in \mathbb{N}$ ).

Values of  $BB$  (size not including accept and reject states):

size	2	3	4	5	6
$BB$	6	21	107	$\geq 47,176,870$	$\geq 7.4 \times 10^{36534}$

### Theorem 29

*The busy beaver function is **not** computable.*

## The Busy Beaver Function is **Not** Computable

Proof: Consider the undecidable language (proved latter)

$$H_{TM}' = \{\langle M \rangle \mid M \text{ is a TM and } M \text{ halts on } \varepsilon\}$$

- Assume **BB** is computable.
- Given  $\langle M \rangle$  compute  $n = \text{BB}(m)$ , where  $m$  is the number of states in  $M$ .
- Run  $M$  on  $\varepsilon$  for  $n + 1$  steps.
- If  $M$  did not halt in  $n + 1$  steps, then it will never halt!



# The **Bounded** Busy Beaver Function **Is** Computable

## Definition 30

For  $d \in \mathbb{N}$ , define the function  $BB_d: \mathbb{N} \mapsto \mathbb{N}$  as

$$BB_d(n) := \begin{cases} BB(n), & n \leq d, \\ 0, & \text{otherwise.} \end{cases}$$

## Theorem 31

*The function  $BB_d$  is computable for every  $d \in \mathbb{N}$ .*

Proof's idea: "Hardwire" the values  $BB(1) \dots, BB(d)$  into a TM to compute  $BB_d$ .



# Reducibility

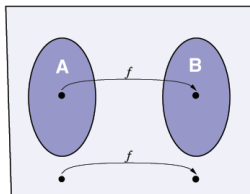
So far, we have seen many examples of **reductions** from one language to another, but the notion was neither defined nor treated formally.

Reductions play an important role in

- decidability theory (here and now)
- complexity theory (to come)

Time to **get formal**.

## Mapping Reductions



### Definition 32

A **computable** function  $f: \Sigma^* \rightarrow \Sigma^*$  is a **reduction** from language  $A$  to language  $B$ , if  $w \in A \iff f(w) \in B$ , for every  $w \in \Sigma^*$ .

If a reduction from  $A$  to  $B$  exists, we say that  $A$  is **mapping reducible** to  $B$ , denoted by  $A \leq_m B$ .

A mapping reduction converts questions about **membership in  $A$**  to **membership in  $B$** .

### Remark 33

Note that  $A \leq_m B \iff \bar{A} \leq_m \bar{B}$

## Applications

### Theorem 34

If  $A \leq_m B$  and  $B$  is decidable, then  $A$  is decidable.

Proof: Let  $M$  be the decider for  $B$ , and  $f$  a (mapping) reduction from  $A$  to  $B$ .

### Algorithm 35 ( $N$ )

On input  $w$

- 1 Compute  $f(w)$
- 2 Emulate  $M$  on input  $f(w)$  and output whatever  $M$  outputs.



### Corollary 36

If  $A \leq_m B$  and  $A$  is undecidable, then  $B$  is undecidable.

In fact, this has been **our** principal tool for proving undecidability of languages other than  $A_{TM}$

## $H_{TM}$ is Undecidable – Revisited

Recall that

- $A_{TM} = \{\langle M, w \rangle \mid M \text{ is a TM that accepts } w\}$
- $H_{TM} = \{\langle M, w \rangle \mid M \text{ is a TM and } M \text{ halts on input } w\}$

Earlier we proved that  $H_{TM}$  undecidable by (de facto) reduction from  $A_{TM}$ . Let's reformulate this.

### Claim 37

$$A_{TM} \leq_m H_{TM}$$

The following **computable** function  $f$  establishes  $\langle M, w \rangle \in A_{TM} \iff f(\langle M, w \rangle) \in H_{TM}$ .

### Definition 38 ( $f$ )

On input  $\langle M, w \rangle$ , return  $\langle M', w \rangle$ .

TM  $M'$  is defined as follows:

On input  $x$ , emulate  $M(x)$  and

**accepts** if  $M$  accepts; **enters a loop** if  $M$  rejects.

## $H_{TM}'$ is Undecidable

Recall that

- $H_{TM} = \{\langle M, w \rangle \mid M \text{ is a TM and } M \text{ halts on input } w\}$
- $H_{TM}' = \{\langle M \rangle \mid M \text{ is a TM and } M \text{ halts on input } \epsilon\}$

### Claim 39

$$H_{TM} \leq_m H_{TM}'$$

The following **computable** function  $f$  establishes

$$\langle M, w \rangle \in H_{TM} \iff f(\langle M, w \rangle) \in H_{TM}'.$$

### Definition 40 ( $f$ )

On input  $\langle M, w \rangle$ , return  $\langle M' \rangle$ .

TM  $M'$  is defined as follows:

On input  $x$ , erase  $x$  and write  $w$  and emulate  $M(w)$ :

**accepts** if  $M$  accepts; **reject** if  $M$  rejects.

## The Mapping Reducible Relation is **Not** Symmetric

$H_{TM}' = \{ \langle M \rangle \mid M \text{ is a TM and } M \text{ halts on } \varepsilon \}$

### Claim 41

Let  $L = \{0^n \mid n \in \mathbb{N}\}$ . Then  $L \leq_m H_{TM}'$ , but  $H_{TM}' \not\leq_m L$

Proof: It is clear that  $H_{TM}' \not\leq_m L$  (why?).

For proving  $L \leq_m H_{TM}'$ , define  $f$  as follows:

### Definition 42 ( $f$ )

On input  $w$ . Return  $\langle M_H \rangle$  if  $w \in L$ , and return  $\langle M_L \rangle$  otherwise.

Where  $M_H$  halts on  $\varepsilon$ , and  $M_L$  loops on  $\varepsilon$ .



# Enumerability

## Theorem 43

*If  $A \leq_m B$  and  $B$  is enumerable, then  $A$  is enumerable.*

Proof is same as before, using accepters instead of deciders.

## Corollary 44

*If  $A \leq_m B$  and  $A$  is **not** enumerable, then  $B$  is **not** enumerable.*

## TM Equality

$$EQ_{TM} = \{ \langle M_1, M_2 \rangle : M_1, M_2 \text{ are TMs and } L(M_1) = L(M_2) \}$$

### Theorem 45

Both  $EQ_{TM}$  and, its complement,  $\overline{EQ_{TM}}$ , are not enumerable.

Stated differently,  $EQ_{TM}$  is neither enumerable nor co-enumerable, or  $EQ_{TM} \notin \mathcal{RE} \cup \mathcal{RE}$ .

We show that

- $A_{TM} \leq_m EQ_{TM}$ , and  $A_{TM} \leq_m \overline{EQ_{TM}}$ .
- It follows that  $\overline{A_{TM}} \leq_m \overline{EQ_{TM}}$  and  $\overline{A_{TM}} \leq_m EQ_{TM}$  (why?)
- Hence, neither  $EQ_{TM}$  nor  $\overline{EQ_{TM}}$  are enumerable.



$$A_{\text{TM}} \leq_m \text{EQ}_{\text{TM}}$$

### Claim 46

$$A_{\text{TM}} \leq_m \text{EQ}_{\text{TM}}.$$

### Definition 47 (f)

On input  $\langle M, w \rangle$

- 1 Construct a machine  $M_1$  that **accepts**  $\Sigma^*$
- 2 Construct a machine  $M_2$  that **accepts**  $x$ , if  $M(w)$  **accepts**.
- 3 Return  $\langle M_1, M_2 \rangle$ .

Note

- if  $M$  accepts  $w$ , then  $M_2$  accepts **everything**. Otherwise,  $M_2$  accepts **nothing**.
- Hence,  $\langle M, w \rangle \in A_{\text{TM}} \iff \langle M_1, M_2 \rangle \in \text{EQ}_{\text{TM}}$ .



$$A_{\text{TM}} \leq_m \overline{EQ_{\text{TM}}}$$

### Claim 48

$$A_{\text{TM}} \leq_m \overline{EQ_{\text{TM}}}.$$

### Definition 49 (f)

On input  $\langle M, w \rangle$ :

- 1 Construct a machine  $M_1$  that **accepts**  $\emptyset$ .
- 2 Construct a machine  $M_2$  that **accepts**  $x$ , if  $M(w)$  **accepts**.
- 3 Return  $\langle M_1, M_2 \rangle$ .

Note

- If  $M$  accepts  $w$  then  $M_2$  accepts **everything**. Otherwise,  $M_2$  accepts **nothing**.
- Hence,  $\langle M, w \rangle \in A_{\text{TM}} \iff \langle M_1, M_2 \rangle \in \overline{EQ_{\text{TM}}}$



## Section 2

# Rice's Theorem

## Non Trivial Properties of $\mathcal{RE}$ Languages

A few examples

- $L$  is finite.
- $L$  is infinite.
- $L$  contains the empty string.
- $L$  contains no prime number.
- $L$  is co-finite.
- ...

All these are **non-trivial** properties of enumerable languages, since for each of them there is  $L_1, L_2 \in \mathcal{RE}$  such that  $L_1$  satisfies the property but  $L_2$  does not.

### Question 50

Are there any **trivial** properties of  $\mathcal{RE}$  languages?

# Rice's Theorem

## Theorem 51

Let  $\mathcal{C}$  be a *proper non-empty subset* of the set of  $\mathcal{RE}$  and let  $L_{\mathcal{C}} = \{\langle M \rangle : L(M) \in \mathcal{C}\}$ . Then  $L_{\mathcal{C}}$  is undecidable.

Proof's idea: Reduction from  $H_{TM}$ .

Given  $M$  and  $w$ , we construct  $M_0$  such that:

- If  $M$  halts on  $w$ , then  $\langle M_0 \rangle \in L_{\mathcal{C}}$ .
- If  $M$  does not halt on  $w$ , then  $\langle M_0 \rangle \notin L_{\mathcal{C}}$ .

## Proving Rice's Theorem

We assume wlg. that  $\emptyset \notin \mathcal{C}$  (otherwise, look at  $\bar{\mathcal{C}}$ , also proper and non-empty).

Fix  $L \in \mathcal{C}$  and let  $M_L$  be a TM accepting it (recall  $\mathcal{C} \subseteq \mathcal{RE}$ ).

### Algorithm 52 ( $M_0$ )

On input  $y$ :

1 Emulate  $M(w)$ .

2 Emulate  $M_L(y)$ :

Accept if  $M_L$  accepts; reject if  $M_L$  rejects.

Let  $f(\langle M, w \rangle) := \langle M_0 \rangle$ , and let  $f(x) = \emptyset$  if  $x$  is not of the form  $\langle M, w \rangle$ .

### Claim 53

$f$  is a mapping reduction from  $H_{TM}$  to  $L_{\mathcal{C}}$

## $f$ is Computable

### Claim 54

$f$  is computable.

Proof: On a valid pair  $\langle M, w \rangle$ , the TM  $M_0 = f(\langle M, w \rangle)$  is simply a concatenation of two known TMs: the **universal machine** and  $M_L$ . ♣

$$\langle M, w \rangle \in H_{TM} \iff f(\langle M, w \rangle) \in L_C$$

### Claim 55

$$\langle M, w \rangle \in H_{TM} \iff f(\langle M, w \rangle) \in L_C$$

(Hence,  $f$  is a mapping reduction from  $H_{TM}$  to  $L_C$ )

Proof:

- If  $\langle M, w \rangle \in H_{TM}$ , then  $M_0$  gets to **Step 2**, and emulates  $M_L(y)$ .  
Hence  $L(M_0) = L \in C$ .  
Otherwise (i.e.,  $\langle M, w \rangle \notin H_{TM}$ ),  $M_0$  never gets to **Step 2**.  
Hence  $L(M_0) = \emptyset \notin C$ .
- Thus,  $\langle M, w \rangle \in H_{TM}$  iff  $\langle M_0 \rangle \in L_C$ .



We proved that  $H_{TM} \leq_m L_C$ , thus  $L_C$  is undecidable.



## Reflections

- Rice's theorem can be used to show **undecidability** of properties like
  - ▶ Does  $L(M)$  contain infinitely many primes
  - ▶ Does  $L(M)$  contain an arithmetic progression of length 15
  - ▶ Is  $L(M)$  empty
- Decidability of properties related to the **encoding** itself **cannot** be inferred from Rice.
  - ▶ The question *does  $\langle M \rangle$  has an even number of states* is decidable.
  - ▶ The question *does  $M$  reaches state  $q_6$  on the empty input string* is **undecidable**, but this **does not** follow from Rice's theorem.
- Rice does **not** say anything on membership in  $\mathcal{RE}$  of questions like *is  $L(M)$  finite*.
- **Rice's Theorem is a powerful tool, but use it with care!**